CIS102 Mathematics for Computing
Examiners' Report 2003
Zone West

The overall performance on this paper was good, and the students are coping well with most of the subject. There were a few poor candidates, but many more that gained extremely high marks.

Many candidates have problems with precise mathematical notation and exact definitions. Thus many did poorly or left blank a question like 6(b) which requires understanding the notation for sets and relations. Frequently it happens also that a candidate knows the answer to a question, but is unable to phrase it using proper mathematical terms. Candidates are advised to pay attention to the notation used in the study guide and to strive to adapt this notation in their own work. When you have solved an exercise, always compare your notation as well as your answers to the model answers provided in the study guide or by your lecturer. If your notation is substantially different, try to improve it.

When revising for the examination, you should attempt the sample examination questions in the study guide. After you have solved them, compare your solutions to the ones given in the guide. It is hard to give a general rule for how you can mark your own answers, as exercises may have more than one correct solution method. However, if you find that you have the right answer, but that your method differs substantially from or is less theoretical than the one offered in the study guide, you might want to reconsider your approach to the solution.

One general mistake made by a large number of candidates is not to show their working and calculations and not explaining the reasoning behind an answer. Especially in questions involving arithmetic, failure to show calculations generally results in the loss of marks. Also questions requiring a short answer are best answered by initially giving a short answer to the question followed by your *reason* for this particular answer. Remember that examiners are more interested in how you arrived at an answer than in the answer itself!

What follows is some answers, hints, solutions and comments on the examination questions that may help you, when you are revising for your examination.

Question 1

(a) The answer is $(101011)_2$.
Many marks were lost here because candidates failed to show the binary working. It is important to show all carries.

(b) The answer is 2610.5. Working must be shown in full.

(c) The answer is $(11010101)_2$.
You must show all the sucessive divisions by 2 with remainders and explain how to get the answer.

(d) Let $y = 0.818181\ldots$. Then $100y = 81.818181\ldots$.
Subtraction yields $99y = 100y - y = 81$, and hence $y = 81/99 = 9/11$.

Question 2

(a) (i) $\{1, 3, 5, 7, 9, 11\}$.

(ii) $\{1/4, 1/2, 1, 2, 4\}$.
The most common error in (ii) was to leave out the two non-integers from the set. Note that it is r which must be an integer, not 2^{r-1}.

(b) (i) Hint: Only one of the 8 regions on the Venn diagram should be shaded.
Many students confuse the two symbols $\cap$ and $\cup$. Try to remember that $\cup$ stands for $\cup$nion.

(ii) $(A \cup B)' \cap C = \{u\}$.
Show your working by first listing the set $A \cup B$ and its complement, hence find the intersection with C.

(iii) $\{r, t, q\} = (A \cup B) - C$ which is most conveniently found by drawing a Venn diagram featuring A, B and C.

Question 3

(a) The truth tables asked for/needed:

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg p \wedge q$	$\neg p \to q$	$p \vee q$
0	0	1	1	1	0	0	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	1	0	1	1

(i) $p \vee \neg q$ is true for $n = 10, 11, 12, 13, 14, 15, 16, 18$.
$\neg p \wedge q$ is true for $n = 17, 19$.

(ii) $\neg p \to q$ is equivalent to $p \vee q$, which can be shown by drawing up truth tables for the two statements and checking that they are equal.

(b) (i) If this apple is red then it is ripe.

(ii) This apple is red and it is not ripe.

(iii) $\neg p \wedge \neg q$

(iv) $\neg p \to \neg q$

Question 4

(a) (i) $f(2) = \lfloor 2/3 \rfloor = 0$, $f(10) = \lfloor 10/3 \rfloor = 3$.

(ii) The range of f is $\mathbb{Z}$.

(iii) The pre-images of 1 are all $x \in \mathbb{R}$ such that $3 \leq x < 6$.
A very common error here was to state that the pre-images of 1 are just 3, 4 and 5. Make sure that you understand that $3 \leq x < 6$ for infinitely many x as f has domain $\mathbb{R}$.

(iv) f is not one-to-one as e.g. $f(2) = f(1) = 0$. Hence f is not invertible as f needs to be both one-to-one and onto to have an inverse.

(b)

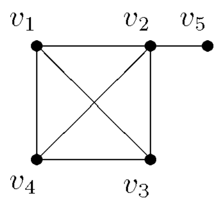
x	1	3	9	27	81	243
$g(x)$	0	1	2	3	4	5
$h(x)$(2 d.p.)	1	1.44	2.08	3	4.32	6.24

Yes, $\log_3 x = O(\sqrt[3]{x})$ since for large values of x we have $\log_3 x < \sqrt[3]{x}$.

Question 5

(a) (i) 4,3,3,3,1.
Note that the degree sequence must be given in either ascending or descending order.

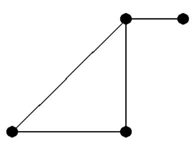
(ii) The graph G:



(iii) $v_3v_2v_1v_4$ and $v_3v_1v_2v_4$ are two paths of length 3 starting at v_3 and ending at v_4. Note that the length of a path is the number of *edges* it contains, many candidates got this question wrong thinking the length of a path is the number of vertices in it.

(iv) E.g. $v_3v_1v_2v_4v_3$.
Note that this question asks for one 4-cycle in G. Many candidates misunderstood the question and listed instead 4 different cycles in G of various lengths.

(b) (i) A graph with degree sequence $3, 2, 2, 1$ exists:



(ii) The sum of the degrees of a graph with degree sequence $4, 3, 2, 2$ is 11, but this is twice the number of edges, and we cannot have 1/2 edge in a graph. Hence a graph with degree sequence $4, 3, 2, 2$ cannot exist.

Many candidates failed to give the complete argument here, stating merely that the sum of the degrees is odd, so the graph cannot exist. It is important to state why it is impossible for a graph to have an odd sum of degrees.

Question 6

(a) (i) Most candidates drew the correct relationship digraph here, though some forgot that all vertices should have loops and some got the direction of the arcs wrong.

(ii) $\mathcal{R}$ is reflexive since for *all* $s \in S$ we have that $s - s = 0$ and so $s\mathcal{R}s$. Note that it is not sufficient to find one s in S that is related to itself when you want to prove that a relation $\mathcal{R}$ is reflexive. You must show that *all* elements of S are related to themselves under $\mathcal{R}$, otherwise $\mathcal{R}$ is not reflexive.

(iii) $\mathcal{R}$ is not symmetric as e.g. $7\mathcal{R}3$ while $3\not\mathcal{R}7$.

(iv) $\mathcal{R}$ is transitive since whenever $s\mathcal{R}t$ and $t\mathcal{R}r$ we have that $s - t = 0, 2$ or 4 and $t - r = 0, 2$ or 4. Thus $s - r = (s - t) + (t - r) = 0, 2, 4, 6$ or 8. But $s - r \leq 7 - 2 = 5$. Hence $s - r = (s - t) + (t - r) = 0, 2$ or 4 and it follows that $s\mathcal{R}r$.

(v) $\mathcal{R}$ is a partial order as it is antisymmetric, reflexive and transitive.
Note that it does not follow from (iii) that $\mathcal{R}$ is antisymmetric. The two properties of symmetry and antisymmetry are not mutually exclusive, a relation may have none of the two properties or it may have both or just one of them.

(b) $\mathcal{R} = \{(000, 3), (001, 2), (010, 2), (100, 2), (110, 1), (101, 1), (011, 1), (111, 0)\}$

Question 7

(a) (i) $u_{n+1} = u_n + n$

$$\begin{aligned} u_1 &= 0; \\ u_2 &= u_1 + 1 = 0 + 1 = 1; \\ u_3 &= u_2 + 2 = 1 + 2 = 3; \\ u_4 &= u_3 + 3 = 3 + 3 = 6; \\ u_5 &= u_4 + 4 = 6 + 4 = 10. \end{aligned}$$

(ii) Many candidates did not attempt this question or did poorly. Many attempts at the proof write down a good base and induction hypothesis, but then go wrong in the induction step because they mix up what they know and what they still need to prove. One good piece of advice, that will save many errors is, that you should never write down an =-symbol without specifically checking that you are absolutely sure you know that what is on the left hand side of it is equal to what is on the right hand side of it.

When proving an identity by induction the proof has 4 steps: the base, the induction hyposthesis, the induction step and a final remark stating why the identity holds by induction.

When proving the base case and the induction step, you must demonstrate that the left hand side (LHS) of the identity is equal to the right hand side (RHS) of the identity for the case in question. It is usually best to keep the two computations completely separate in order not to confuse what you know and what you have not proven yet.

Here we want to prove the identity $u_n = \frac{n(n-1)}{2}$ for all $n \geq 1$:

Base case: When $n = 1$, $LHS = u_1 = 0$ and $RHS = \frac{1(1-1)}{2} = 0$. The identity thus holds for $n = 1$.

Induction hypothesis: Suppose the identity holds for some $k \geq 1$, i.e.

$$u_k = \frac{k(k-1)}{2}.$$

Induction step: We must prove that the identity also holds for $n = k+1$:

$$RHS = \frac{(k+1)((k+1)-1)}{2} = \frac{(k+1)k}{2} = \frac{(k^2+k)}{2}.$$

$$\begin{aligned} LHS &= u_{k+1} \\ &= u_k + k && \text{by the recurrence relation} \\ &= \frac{k(k-1)}{2} + k && \text{by the induction hypothesis} \\ &= \frac{k^2-k}{2} + \frac{2k}{2} \\ &= \frac{k^2+k}{2} \\ &= RHS. \end{aligned}$$

Hence the result is also true for $n = k+1$ and thus for all $n \geq 1$ by induction.

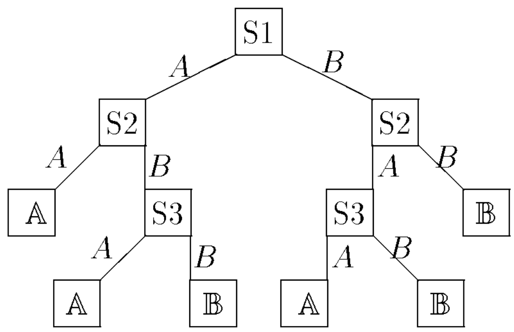
(b) $1 + 4 + 7 + 10 + \ldots + (3n-2) = \sum_{i=1}^{n}(3i-2)$.

Many candidates got this question wrong because they used n as the variable in the sum instead of i. You cannot use n as the variable as the last term of the sum is $(3n-2)$, and n is thus a constant in the context of this exercise.

$$\sum_{i=1}^{100}(3i-2) = 3\sum_{i=1}^{100} i - \sum_{i=1}^{100} 2 = \frac{3(100)(101)}{2} - 200 = 14950.$$

Question 8

(a) A binary tree modelling the outcomes of the match could look like this:



Many marks were lost because candidates failed to explain their models. If you are asked to draw a graph to model something, it is very important that you explain clearly what your vertices and edges represent and how the reader is supposed to get the required information from the graph.
Here the internal nodes S1, S2 and S3 depict the 3 sets, the labels on the edges are the winners of the set above the edge. A path from the root S1 to a leaf represents one possible match. The label on a leaf denotes who wins the match represented by the (unique) path from the root to that leaf.

$P(A \text{ wins in 3 sets}) = 3/5 \times 2/5 \times 3/5 + 2/5 \times 3/5 \times 3/5 = 36/125.$

$P(\text{ match lasts 2 sets}) = 2/5 \times 2/5 + 3/5 \times 3/5 = 13/25.$

(b) Hint: The root of the tree is 8 and the two nodes at level 1 are 4 and 12.
When making a binary search tree, always remember the terminating nodes, the records are to be stored at the internal nodes only. The height of a tree is the (maximum) number of *edges* in a path from the root to a leaf. The tree in this exercise has height 4, and the maximum number of comparisons a computer would have to make in order to match an existing record is thus 4.

Question 9 Many candidates scored very poorly on this exercise in combinatorics and statistics, we thus give the solutions in full.

(a) $10 \times 9 \times 8 \times 7 = 5040$. Here we just used the multiplication principle, the first digit in the PIN can be any one of the 10 digits, including 0. Repetition is not allowed, so there are just 9 candidates for digit 2 of the PIN, 8 candidates for digit 3 and 7 candidates for digit 1.

(b) A PIN is odd if its 4th digit is odd, so we have 5 choices for digit 4. After choosing digit 4, digits 1,2 and 3 can be chosen freely among the remaining 9 digits (without repetition).
Hence $5 \times 9 \times 8 \times 7 = 2520$ PINs are odd.

A PIN ≥ 8000 starts with either 8 or 9, digits 2, 3 and 4 can be chosen freely among the remaining 9 digits (without repetition).
Hence $2 \times 9 \times 8 \times 7 = 1008$ PINs are ≥ 8000.

$1 \times 8 \times 7 \times 5$ PINs start with the number 8 and end in an odd digit,
$1 \times 8 \times 7 \times 4$ PINs start with the number 9 and end in an odd digit (as 9 may not be repeated there are only 4 choices for digit 4), hence $1 \times 8 \times 7 \times 5 + 1 \times 8 \times 7 \times 4 = 504$ PINs are both odd and ≥ 8000.

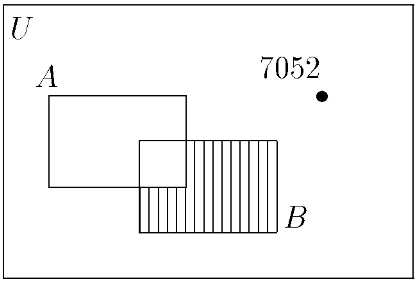
(c) Using the figures computed in (b) and (a) we have

(i) $P(A) = 2520/5040 = 1/2$.

(ii) $P(B) = 1008/5040 = 1/5$.

(iii) $P(A \cap B) = 504/5040 = 1/10$. Note that many candidates lost the mark here because they offered the wrong solution $P(A \cap B) = P(A)P(B) = 1/2 \times 1/5 = 1/10$. This solution is wrong even though it accidentally gives the correct figure in the end, because the formula $P(A \cap B) = P(A)P(B)$ can only be used when it is known that A and B are independent events. We have not proven this, so you may not assume that A and B are independent.

(d) Here the universal set U is the set of PINs. The final Venn diagram looks like this:



Question 10

(a) To show that $AB \neq BA$ you just need to compute the two products and see that they are not equal.

(b) (i) $C = A - B$

(ii) Note that when doing matrix arithmetic, we can add and substract matrices (of the same shape), we can multiply matrices (of suitable shape), but we have not defined any division operation for matrices. In spite of this, a large number of candidates offered as their answer in this question that $D = B/B = 1$. This is a grave error.

$$D = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) $E = B - 2A$.

(c) Many candidates do not know the difference between the augmented matrix for a system of equations and writing the system as a matrix equation, and they thus lost a mark here. The augmented matrix for the system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 3 \end{array}\right)$$

The solution is $(x_1, x_2, x_3) = (1, 1, 2)$.
Note that you are required to solve the system using Gaussian Elimination, hence you must clearly demonstrate that you have done so, and any other solution method gets little or no credit. It is also a good idea to write down all the elementary row operations you do on the augmented matrix.