THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALLS

UNIVERSITY OF LONDON **291 0316 ZA**

BSc Examination
for External Students

COMPUTING AND INFORMATION SYSTEMS AND CREATIVE COMPUTING

Mathematical Techniques of Operational Research

Dateline: Tuesday 5 May 2009 : 2.30 – 4.45 pm

Duration: 2 hours 15 minutes

There are five questions on this paper. Candidates should answer **FOUR** questions only. There are 100 marks available on this paper.

A hand held calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics, texts or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

GRAPH PAPER is required for this examination.

UL09/906

Q1. Designers of wood block puzzles for children, have ideas for four new puzzles P_1, P_2, P_3 and P_4, involving combinations of three different basic blocks, B_1, B_2 and B_3.

The production and expected profit each week is to be based on the constraints that the maximum number of blocks available is 4320 of B_1, 4860 of B_2 and 5670 of B_3.

A preproduction mathematical model of the project is based on the following table which shows the number of blocks required for each puzzle and the expected profit on its sale.

	P_1	P_2	P_3	P_4
B_1	12	0	6	10
B_2	6	18	6	10
B_3	6	9	12	8
Profit *cents*	63	70.5	72	74

The problem for the designers can be modelled by the following linear programming problem:-

Find $x_1, x_2, x_3, x_4 \in \mathbb{R}$ to maximize $z = 63x_1 + 70.5x_2 + 72x_3 + 74x_4$
subject to :-

$$
\begin{array}{rcll}
12x_1 + 0x_2 + 6x_3 + 10x_4 & \le & 4320 & (1) \\
6x_1 + 18x_2 + 6x_3 + 10x_4 & \le & 4860 & (2) \\
6x_1 + 9x_2 + 12x_3 + 8x_4 & \le & 5670 & (3) \\
x_1, x_2, x_3, x_4 & \ge & 0 & (4)
\end{array}
$$

(a) For the linear programming problem above [3]
 (i) define the decision variables x_1, x_2, x_3, x_4,
 (ii) explain how the objective function z is derived.

(b) Construct the initial tableau for the solution of this problem by the Simplex algorithm and state the augmented initial basic feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$. [3]
[Do not perform any iterations on this initial tableau.]

(question continues on next page)

(c) Applications of the Simplex algorithm lead to the following tableau in which x_5, x_6 and x_7 are the slack variables.

Eqn	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	*RS*
0	1	0	0	0	0	$7/3$	2	$23/6$	41535
1	0	$5/4$	0	0	1	$1/8$	$1/24$	$-1/12$	270
2	0	$-1/3$	1	0	0	$-1/18$	$1/18$	0	30
3	0	$-1/12$	0	1	0	$-1/24$	$-5/72$	$5/36$	270

(i) Obtain the objective function z given by the above tableau, and use your expression for z to explain why the maximum value of z is 41535. [3]

(ii) Given that $x_1 = 12r$, $r \in \mathbb{R}, r \geq 0$, state the resulting values of the decision variables x_2, x_3 and x_4. Also, find the range of values r for which this problem has multiple solutions. [5]

(iii) Use $x_1 = 12r$, and the resulting values for x_2, x_3 and x_4 found in part (ii), to substitute in the objective function given by $z = 63x_1 + 70.5x_2 + 72x_3 + 74x_4$ and show that z is independent of r. [4]

(iv) Find the maximum and minimum total number of puzzles that can be produced for the same optimum profit of \$415.35. Also state how many blocks are unused in each case. [3]

(d) The designers now require answers to the following two suggestions.

(i) Suppose the number of blocks available was halved to 2160, 2430, 2835 for B_1, B_2, B_3 respectively. What would be the new values of x_2, x_3, x_4 with $x_1 = 48$, and what would be the optimal profit? [2]

(ii) Suppose the work is to take two weeks instead of one week. The number of blocks available remains unchanged at 4320,4860,5670 for B_1, B_2, B_3 respectively and $x_1 = 48$ but, the profit on puzzle P_3 is increased from 72 cents to $(72 + w)$ cents, where $w > 0$. What would be the new profit for the two weeks work? [2]

Q2. Two players A and B, play a zero sum game in which A has strategies A_1, A_2, A_3, A_4, A_5 and B has strategies B_1, B_2, B_3, B_4, B_5.

The payoff to A, when A plays strategy A_i and B plays strategy B_j, is given by the element (d_{ij}) in the following game matrix D_1.

matrix D_1 =

	B_1	B_2	B_3	B_4	B_5
A_1	4	−2	−1	0	4
A_2	−1	2	0	5	6
A_3	1	3	2	1	3
A_4	0	2	1	0	−1
A_5	3	−3	−1	−1	5

(a) For this game, find [3]

(i) the safest single strategy for each player,

(ii) the guaranteed minimum gain $\underline{v}$ to A,

(iii) the guaranteed maximum loss $\overline{v}$ to B.

(iv) State the connection of $\underline{v}$ and $\overline{v}$ with the value v of the game.

(b) (i) State both sets of inequalities that are used to see if one strategy dominates another strategy in any payoff matrix for a zero sum game between two players. [4]

(ii) Explain how the matrix D_1 in part (a) may be reduced to the

$$\text{matrix } D_2 = \begin{bmatrix} 4 & -2 & -1 & 0 \\ -1 & 2 & 0 & 5 \\ 1 & 3 & 2 & 1 \end{bmatrix}$$

(c) Using the reduced matrix D_2 of part (b), suppose that A plays a mixed strategy $\mathbf{p} \in \mathbb{R}^3$ with a minimum expected gain of v_2, against each of B's strategies, while B plays a mixed strategy $\mathbf{q} \in \mathbb{R}^4$ with a maximum expected loss of v_1, against each of A's strategies. Describe how the problem, of finding the optimal values of $\mathbf{p}$ and $\mathbf{q}$, can be formulated as a pair of dual linear programming problems in new variables $\mathbf{y}$ and $\mathbf{x}$. Explain why the second row of the matrix D_2 is so important for the formulation. [7]

(question continues on next page)

(d) Using the payoff matrix D_2, construct the initial tableau to solve the reduced problem by the Simplex method. [4]
[Do not perform any iterations on this initial tableau.]

(e) The final tableau, for the solution of this linear programming formulation of B's problem in the reduced game, is given below, where x_5, x_6 and x_7 are the slack variables. [7]

Eqn	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	*RS*
0	1	0	$^{14}/_{23}$	0	0	$^{3}/_{23}$	$^{2}/_{23}$	$^{13}/_{23}$	$^{18}/_{23}$
1	0	1	$^{-7}/_{46}$	0	0	$^{13}/_{46}$	$^{5}/_{23}$	$^{-1}/_{46}$	$^{7}/_{23}$
2	0	0	$^{-17}/_{46}$	0	1	$^{1}/_{23}$	$^{9}/_{46}$	$^{1}/_{46}$	$^{6}/_{23}$
3	0	0	$^{32}/_{23}$	1	0	$^{-3}/_{23}$	$^{-2}/_{23}$	$^{10}/_{23}$	$^{5}/_{23}$

(i) From the above tableau find the value of the game, and **p** and **q**. Also find the maximum possible number of points that player B could gain, when playing strategy B_3, if both players follow the strategies given by **p** and **q** when playing the game 1656 times.

(ii) Suppose that the reduced game matrix D_2 has each and every element increased by 2 to become the new game matrix given by

$$\text{matrix } D_3 = \begin{bmatrix} 6 & 0 & 1 & 2 \\ 1 & 4 & 2 & 7 \\ 3 & 5 & 4 & 3 \end{bmatrix}$$

State the value of the new game and the new values of x_1, x_3, x_4.

Q3.(a) Consider the following linear programming problem **P**.

Find $x_1, x_2, x_3 \varepsilon \mathbb{R}$ to maximize $z = 9x_1 + 3x_2 - 4x_3$
subject to :-

$$\begin{aligned} 7x_1 + 4x_2 - 2x_3 &\leq 812 \quad (1) \\ -4x_1 + 3x_2 + 6x_3 &\geq 114 \quad (2) \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(i) By introducing, ***and identifying***, slack, surplus and artificial variables, prepare the above constraints (1) and (2) for the solution of **P** by the Simplex algorithm. [3]

(ii) Using the equation $z_0 = z - M\bar{x}_6$, modify the objective function z for solution by the Big M method. [3]

(iii) Complete the solution of **P** by the Big M method. State the values of x_1, x_2, x_3 and z in the optimal solution to **P**. [7]

[All entering and leaving variables must be identified in the iterations.]

(b)(i) Use a graphical method to find both the ***minimum and maximum*** values of $z = 7y + 3x$ subject to the following constraints:- [10]

$$\begin{aligned} -10 \leq 3y - 2x &\leq 14 \\ y + 2x &\leq 34 \\ y &\leq 10 \\ x &\geq 3 \end{aligned}$$

Graph paper **must** be used with a scale of 10 mm to 1 unit for both x and y axes. A value line $z = 21$ **must** be drawn, and the feasible region clearly identified by lettering the vertices of the region A,B,C,D,E,F.

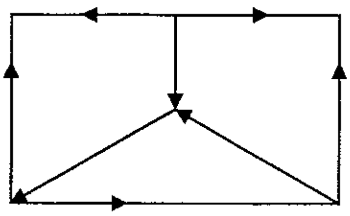
(Note: The correct graphical work will be contained in the first quadrant by marking each axis from 0 to +16.)

(ii) Find the **minimum and maximum** value of $z = 3y - 7x$ subject to the same constraints in part (b)(i) above. [2]

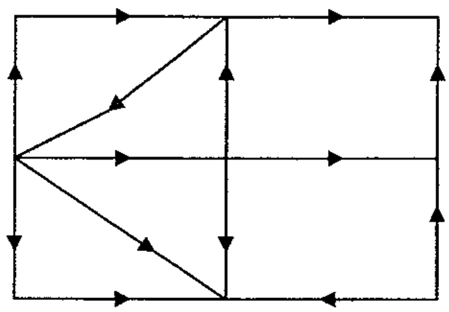
Q4.(a)(i) Prove that every acyclic directed network contains a source. [6]

(ii) Given that N is an acyclic directed network, prove that the vertices of N can be given an acyclic ordering. [6]

(iii) Show, and explain, why Network 1 below is not acyclic and the connection between this network and the proof in part (i). [3]



Network 1



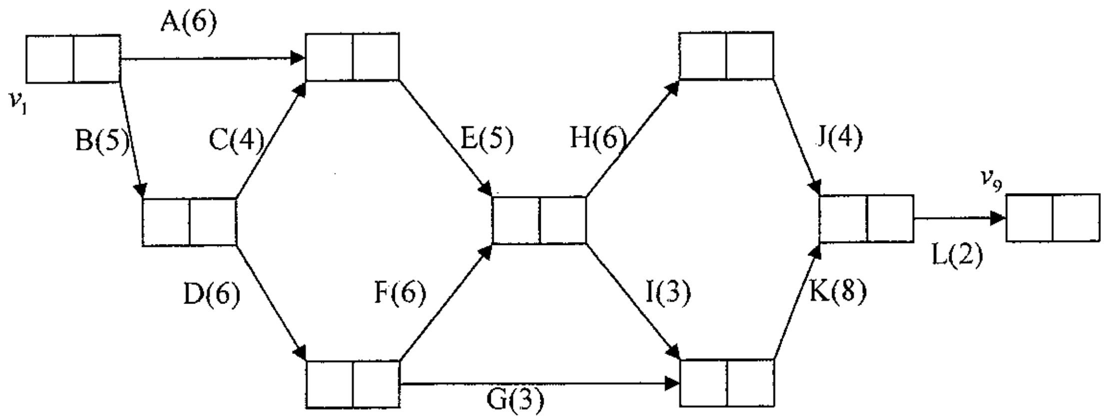
Network 2

(iv) Network 2 above is not acyclic but, can be made acyclic by changing the direction of one and only one of the arcs. Find this arc, change its direction, and then, give the network an acyclic ordering $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$. [4]

(b) The table below describes a network $\boldsymbol{P}$ with five vertices v_1, v_2, v_3, v_4, v_5. Without stating Prim's algorithm, use the algorithm to find a minimum weight spanning tree in $\boldsymbol{P}$. [6]

	v_1	v_2	v_3	v_4	v_5
v_1		7	3	8	5
v_2	7		5	3	6
v_3	3	5		4	7
v_4	8	3	4		8
v_5	5	6	7	8	

Q5.(a) The C.P.A.-network below is an initial exercise to examine the work sequence required to possibly complete a contract within three days. There are to be twelve essential activities A,B,C,D,E,F,G,H,I,J,K,L. Each arc of the network is lettered to denote an activity and show the number of hours required to complete that activity using only one worker.



(i) Define the early event time $ET(v)$ and the late event time $LT(v)$ for an event v in a C.P.A.-network. [2]

(ii) Draw the C.P.A.-network for the contract and give your network an acyclic labelling $v_1, v_2, \ldots, v_9$ where v_1 represents the starting event of the work. Also insert the $ET(v)$ and $LT(v)$ for each event. Show that the shortest completion time is 30 hours. [6]

(iii) Given that an activity is represented by the arc $v_i v_j$ with weight $w(v_i v_j)$, state the formula for the float time of this activity. Tabulate the total float time for all twelve activities and identify any possible critical paths on your copy of the network. [3]

(iv) Assume that two workers, W_1 and W_2, would agree to work ten hours a day. By obtaining a three day work timetable for each worker show that they could complete the contract in 30 hours. [3]

(question continues on next page)

(v) In an attempt to see if the contract can be completed in three days by working nine hours a day it is suggested that each of the activities F,H and K should be reduced by 2 hours. [4]
Enter the new ET(v)s and LT(v)s on your network (Put numbers above or below the original entries.) and show that the new shortest completion time is 26 hours.
State the last three hours (24,25 and 26) of both new work timetables.

(b) Draw the C.P.A.-network for the following project with the activities P,Q and R starting from the same event at the beginning of the project. [7]

The network must be drawn with only ***seven*** events and using ***two and only two dummy activities*** to show that the shortest completion time for the project is 24 hours.

Activities	*Duration(hours)*	*Preceding Activities*
P	8	
Q	9	
R	10	
A	6	Q
B	10	P,Q
C	5	R,A
D	4	B,C

Give the network an acyclic labelling $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ and enter every ET(v) and LT(v).

END OF EXAMINATION