Chapter 1

Introduction

1.1 Relationship to previous mathematics units

The pre-requisite Mathematics for this unit is **05a Mathematics 1** and **05B Mathematics 2** (or their equivalents if you have been awarded an exemption) and **116 Abstract mathematics**.

You should therefore have knowledge and understanding of basic linear algebra and computational aspects of calculus with regard both to single variable integration and to the basic essentials of multiple-variable differential calculus with optimisation in mind, including the use of Lagrange Multipliers. At some critical points later in the unit you will need to rely on a facility to perform calculations with integrals, so we highlight the level of difficulty in some learning activities in Chapter 2. Chapter 8 relies on knowing how to define the Riemann integral, a matter that is covered in either of the units **116 Abstract mathematics** or **117 Advanced calculus**. Elementary notions of Probability or Statistics are also depended upon.

In most of the unit **43 Mathematics of finance and valuation** we use the language of Linear Algebra. This does of course mean that we will also be guided by the usual geometric arguments that are associated with ideas from Linear Algebra. That is so even when the arguments are motivated by probability. Note that the approach taken in this unit does not rely on any use of measure theory.

We will think of the component of a vector as describing various possible sums of money payable according to future prevailing circumstances. That is, if $x = (x_1, ..., x_n)$ is a vector in $\mathbb{R}^n$, we will want to think of that vector as referring to a model of future events arising as n different possible circumstances, labelled $1, 2, ..., n$ and called 'the states of nature'. Then the component x_i is interpreted as a sum of money payable when the state of nature is i. Thus x models what is known in Probability Theory as a 'random variable' X with the states of nature described by the sample space $\Omega = \{1, 2, ..., n\}$ and the realization of X when the state of nature is i is x_i.

In Chapter 2 we help you to line up your assumed knowledge of Linear Algebra with your assumed knowledge of Basic Probability and Statistics. This is an important step which enables us to use geometric intuition to solve problems originating in the valuation of financial contracts.

By financial contract we mean reasonably standard financial arrangements such as insurance policies, or a contract for the execution of building works which may involve foreseeable but currently unknown complications. In all these we ask how much to pay 'today' for a contractually specified payment (albeit uncertain sum of money) to be received at some specified future date such as the proverbial 'same time next year'.

The aim of this unit is to identify the fundamental concepts and methods of Financial Mathematics. We will thus learn two approaches to representing the uncertain evolution of asset prices, first in discrete time and then in continuous time. We will formulate a basis for valuing well-defined future payments that depend on one of a number of specified circumstances occurring when it is not known in advance which of these circumstances will arise.

The mathematical arguments in discrete time will be conducted rigorously, meaning that terms will be precisely defined and results will be proved. Of course plenty of motivation and informal explanations will be given.

We will rely on the rigorous development in discrete time to support a much more informal approach to the continuous time approach. In the second part (the continuous time context) the emphasis will be on using calculus to obtain valuations of what are called European call and put options.

So, for this subject, you will not only have to solve problems: you will have to be able to reason abstractly, and be able to **prove** or **justify** things. In most of this subject, we need to work with **precise definitions**, and you will need to know these. Not only will you need to know these, but you will have to understand them, and be able (through the use of them) to demonstrate that you understand them. Simply learning the definitions without understanding what they mean is not going to be adequate. One hopes that these words of warning won't discourage you, but it's important to make it clear that this is a subject at a higher level than some of its prerequisites.

1.2 Aims of the unit

This subject is intended to enable students to:

- acquire fundamental skills in the techniques of mathematical finance
- understand the principles underlying the subject of mathematical finance
- prepare for further units in mathematical finance and/or related disciplines (e.g. economics, actuarial science).

1.3 Learning objectives

Having followed this unit, students should have:

- knowledge, understanding and formulation of the principles of risk-neutral valuation including some versions of the No-Arbitrage Theorem
- knowledge of replication and pricing of contingent claims in certain simple models (discrete and continuous)
- knowledge of the derivation of the Black-Scholes equation, its solution in special cases, the Black-Scholes formula.

Having completed this unit, and the relevant readings, you should be able to:

- demonstrate knowledge of the subject matter, terminology, techniques and conventions covered in the subject
- demonstrate an understanding of the underlying principles of the subject
- demonstrate the ability to solve problems involving understanding of the concepts.

1.4 Syllabus

This is an introduction to an exciting and relatively new area of mathematical application. It is concerned with the valuation (i.e. pricing) of 'financial derivatives'. These are contracts which are bought or sold in exchange for the promise of some kind of payment in the future, usually contingent upon a share-price then prevailing (of a specified share) or alternatively the level achieved by a share index, i.e. by a certain weighted average of share prices. They are called 'derivatives' because they are **derived** from some **underlying** financial asset such as a share.

The unit reviews the financial environment and some of the financial derivatives traded on the market. It then introduces the mathematical tools which enable the modelling of the fluctuations in share prices. Inevitably these are modelled by equations containing a random term. It is this term which introduces risk; it is shown how to counterbalance the risks by putting together portfolios of shares and derivatives so that risks temporarily cancel each other out and then this strategy is repeated over time. As this procedure resembles hedging a bet – i.e. betting both ways – one talks of dynamic hedging. A very intuitive valuation argument (albeit now regarded as only of historical interest, see Section 9.7 for details) is based on 'hedging' and begins like this: the yield of a temporarily riskless portfolio must equal the rate of return offered by a safe deposit bank account (i.e. a riskless bank rate). The latter, of course, needs to be assumed to exist. This equation assumes that the market which values shares and derivatives actually is in equilibrium and hence eliminates the opportunities of 'arbitrage' (such as making a sure-thing profit from, say, buying cheap and selling dear).

The 'riskless hedge' argument, just mentioned, implies in the continuous-time model that the price of a derivative is the solution of a differential equation. One may either attempt to solve the differential equation by standard means such as numerical techniques or via Laplace transforms, though this is not always easy or feasible. However, there is an alternative route which may provide the answer: a calculation of the expected payment to be obtained from the contract by using what is known as the synthetic probability (or the risk-neutral probability). One proves that, regardless of what an investor believes the expected growth rate in the share price will be, the dynamic hedging has the effect that he might as well believe the growth rate to be the riskless growth rate. Though this may seem obvious in retrospect it does require some careful reasoning to justify.

The unit considers two approaches to risk-neutral calculation, using discrete time and using continuous time. Continuous time requires the establishment of a second-order volatility correction term when using standard first-order approximation from calculus. This leads to what is known as the Itô Rule/Formula. Finite time arguments need some apparatus from Linear Algebra such as the Separating Hyperplane Theorem. We enter the subject from the discrete-time model for an easier discussion of the main issues.

1.5 Organisation of the subject guide

After this introductory chapter, in which we also discuss the reading list, the next chapter recalls some basic background in mathematics. Chapter 3 introduces you to some financial ideas with which the later mathematics will be concerned.

The unit proper begins, in effect, with Chapter 4, which is dedicated to a study of a very simple framework. There is just one risky asset and money ('cash' to be precise), just one future date, and only two possible 'states of nature' that might occur. One state is such that an economic agent may regard it as 'favourable' for himself and another that he regards as 'unfavourable'. Of course another economic agent may view these two states the other way about. What matters is that there are only two states. The aim is to explain two central ideas. One is 'replication' of a claim by means of a portfolio, which enables the claim to be valued by the cost of purchasing the portfolio. This means that once this cost is incurred the holder of the portfolio is not exposed to any risks associated with settling his future liabilities. He is made neutral to risk. The other idea is valuation using expected values with reference to a 'purpose-made' probability measure called a 'risk-neutral' measure which is sometimes referred to as the 'synthetic probability' (for an emphasis of its specific purpose). Thus the expected value of the claim under this measure is to agree with the purchase cost of the replicating portfolio.

Once understood in the simplest concept, we re-establish in Chapter 5 the two central ideas in a one-period model with several assets and several states. We prove the Fundamental Theorem of Asset Pricing, namely that a risk-neutral measure exists if and only if there are no arbitrage opportunities.

In Chapter 6 the two ideas are extended to cover multiple periods aided by the idea of a 'self-financing' trading strategy. This is intellectually the hardest part of the unit as it uses ideas from several quarters. The main tool is conditional expectation and this requires quite a lot of technical apparatus to define rigorously what information is known at certain points in time.

Whereas Chapter 6 assumes a quite general framework for the evolution of asset price, we discuss in Chapter 7 a specific approach to modelling asset price, the Binomial model. In this model, price changes are generated at each period in a uniform way by two constant factors, one factor inducing a possible price move up, the other an alternative price move down. This uniform behaviour permits easy formulas for valuation of European calls and puts. Furthermore it offers easy computation procedures for evaluating American puts. The significance of the model comes from limit considerations. In an appropriately constructed limit (as the number of periods tends to infinity and the time between periods tends to zero) the model's European call and put valuations tend to the corresponding valuations in the Black-Scholes continuous-time model.

The theme of Chapter 8 is modelling price uncertainty in continuous time. As its base the model assumes an anticipated (constant) rate of growth μ for the price of an asset and a noise of constant 'amplitude' σ. The growth rate is modelled along the lines of the deterministic interest rate reviewed in Chapter 3. Thus the anticipated percentage growth in price over a time span of Δt is $\mu \Delta t$. The further ingredient in the model is 'standard' noise as generated by a 'stochastic process', that is by a family of random variables indexed by time, denoted here z_t. The standard amount of noise over the time span Δt starting at time t is Δz_t, which is defined to be $z_{t+\Delta t} - z_t$. It is assumed that Δz_t is normally distributed with variance Δt. The noise added to the anticipated growth is then $\sigma \Delta z_t$, so that the 'standard noise' is amplified or attenuated by a factor σ. Thus the noise element has variance $\sigma^2 \Delta t$, which is thus proportional to the time span Δt. The central tool developed in this Chapter is Itô's Formula. This is concerned with functions of price and allows the computation of the increment in a differentiable function when the stochastic price changes over a time interval. The Formula gives a stochastic form of Taylor's Theorem.

The techniques of Chapter 8 are put to use in the Chapter 9. Time is discretized so that the time span between periods is fixed but arbitrarily small, along lines familiar in units **116 Abstract mathematics** or **117 Advanced calculus**. The Itô Formula is used to approximate increments in value. Valuation proceeds by the familiar route of replication over the individual periods between the discrete time points. The emphasis is on deriving in the limit (as the time spans tend to zero) a valuation of a European option contract written on some asset. This will be a deterministic function which identifies the option value in terms of two free variables: the current asset price, and the time left to the expiry date of the contract. The value function may thus be interpreted as the conditional expected value of the option pay-off under a risk-neutral probability with conditioning on the current asset price and the time to expiry. We are able to identify what this risk-neutral measure is once we derive the celebrated Black-Scholes equation. This is the partial differential equation satisfied by the value function, and we obtain it from a consideration of increments in value. So we are finally able to identify the risk-neutral measure. Armed with the risk-neutral measure we can value a European call option.

The final chapter is concerned with American long-dated options, that is, ones which have a relatively long life left (their expiry date is 'far' into the future). For the purposes of a tractable analysis of such options we place their expiration date at infinity. Options which expire at infinity are called perpetual. They satisfy an ordinary differential equation version of the Black-Scholes equation and the equation may be solved explicitly. Thus we are able to trace the graphs of perpetual options and have an indication of the value of a long-dated American option.

Finally, an epilogue sums up, in slightly more technical jargon, what we hope you will have achieved by the end of this unit.

Not all chapters of the guide are the same length. It should not be thought that you should spend the same amount of time on each chapter. I will not try to specify how much relative time should be spent on each: that will vary from person to person, and we do not want to be prescriptive. I can however indicate roughly what proportion of time might be spent by a lecturer teaching the material in this guide. A lecturer would judiciously pick which examples to include in a lecture, and which to leave to the students to study on their own. Though the mathematics of subsequent chapters is not any harder in this unit than in any other, the table below articulates the significant need to spend time in Chapter 3 on building up an understanding of the financial environment.

Chapter 2	Vector space approach	5%
Chapter 3	Financial environment	20%
Chapter 4	One risky asset and two states	10%
Chapter 5	One period many assets	10%
Chapter 6	Multi-period models	20%
Chapter 7	The binomial model	15%
Chapter 8	Continuous-time modelling	5%
Chapter 9	Black-Scholes model	10%
Chapter 10	Perpetual options	5%

1.6 Reading advice

Notes on the reading lists

Most topics in this unit are covered in great detail by a large number of books. For that reason, we have resisted the temptation to specify **essential** reading in each chapter of the guide. What is true, however, is that **textbook reading is essential**. Textbooks will provide more in-depth explanations than you will find in this guide (which is explicitly not meant to be a textbook), and they will also provide many more examples to study, and many more exercises to work through.

The following books are the ones I have referred to in this guide, listed roughly in order of usefulness and grouped according to their level. In the first group, the first text, by Pliska, is one of the earliest textbooks on the 'undergraduate market'. Our treatment in Chapters 5 and 6 is inspired in part by Pliska. The first Shreve text is a very accessible alternative for the core material of Chapters 4–6. We follow the notation established in Pliska with two exceptions. Firstly, when considering a trading strategy or portfolio H we write $V_t(H)$ for its time t value rather than V_t. Secondly, we refer to Pliska's 'dominant strategies' as 'sure-thing arbitrage strategies'. We strongly recommend that the student reads through the book by Hull to understand the institutional arrangements governing stock-exchanges and the contracts that they issue. A very good second text for this material is Cvitanić and Zapatero. It is hard to be quite as prescriptive in relation to the material of Chapters 7–10 because our approach is informal – thus we describe, rather than define, stochastic integrals as limiting sums avoiding the requisite formalities. Hull contains an account of the informal theory. Luenberger is exquisite in clarity. Baxter and Rennie is admirable on this front, though written for the 'practitioners' in the finance houses. That has been replaced by its 'classroom' version, a more technically demanding text, by Alison Etheridge aimed at university students with measure theory as a prerequisite. The second Shreve text is accessible but more formal. Finally, for a good background for linear algebra and advanced calculus see the book by Ostaszewski (which is the recommended text for unit 117), or Binmore

Notation variant to Pliska's

and Davies.

The second group of books on the reading list is meant for a very special reader: students should know by now that any unit such as this is an introduction to the collective knowledge amassed by an army of mathematicians, so these books are a pointer for ambitious students looking for further developments. The books in the third group are popularisations and provide either anecdotal background or a historical perspective.[1]

[1] Earlier editions than those listed here are equally useful.

Recommended reading

Shreve, S. *Stochastic Calculus for Finance I, The Binomial asset pricing model.* (Springer, 2004) [ISBN 978-0387249681].

Shreve, S. *Stochastic Calculus for Finance II, Continuous-time models.* (Springer, 2004) [ISBN 978-0387401010].

Pliska, S.R. *Introduction to Mathematical Finance - Discrete Time Models.* (Blackwell, 1998) [ISBN 978-1557869456].

Hull, J.C. *Options, Futures and other Derivative Securities*. (Prentice Hall, 2005) sixth edition [ISBN 978-0131499089]. See also the URL: http://www.rotman.utoronto.ca/~hull/

Cvitanić, J. and F. Zapatero *Introduction to the Economics and Mathematics of Financial Markets.* (MIT, 2004) [ISBN 978-0262033206, 978-0262532594 (solutions manual)].

Luenberger, D. *Investment Science.* (Oxford University Press, 1997) [ISBN13: 9780195108095].

Roman, S.*Introduction to Mathematics of Finance. Undergraduate Texts in Mathematics.* (Springer, 2004) [ISBN 978-0387213644].

Baxter, M. and A. Rennie *Financial calculus.* (Cambridge University Press: Cambridge, 1996) [ISBN 978-0521552899].

Etheridge, A. *A Course in Financial Calculus.* (Cambridge University Press: Cambridge, 2002) [ISBN 978-0521890779].

Ostaszewski, A. *Advanced Mathematical Methods*. (Cambridge University Press: Cambridge, UK, 1991) [ISBN 978-0521289641].

Binmore, K. and J. Davies *Calculus: Concepts and Methods.* (Cambridge University Press: Cambridge, UK, 2001) [ISBN 978-0521775410].

Intermediate reading

Bingham, N.H. and R. Kiesel *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives.* (Springer, 1998) first edition [ISBN 1852330015] second edition, 2004 [ISBN 978-1852334581].

Campbell, J.Y., A.W. Lo and A.C. MacKinlay *The econometrics of financial markets*. (Princeton University Press) [ISBN 978-0691043012].

Janson, S. *Gaussian Hilbert Spaces.* Cambridge Tracts in Mathematics

129 (Cambridge University Press, 1997) [ISBN 978-0521561280 (hbk)] in connection with the geometric view of probability sketched in Chapter 2.

Williams, D. *Probability with martingales.* (Cambridge, 1991) [ISBN 978-0521406055].

Øksendal, B. *Stochastic differential equations.* (Springer, 1998) [ISBN 978-3540047582].

Wilmott, P., S. Howison and J. Dewynne *Mathematics of Financial Derivatives.* (Cambridge University Press: Cambridge, 1995) [ISBN 978-0521497893].

Whittaker, E.T. and G.N. Watson *A course of modern analysis.* (Cambridge University Press, 1984) [ISBN 978-0521588072].

Durrett, R. *Stochastic Calculus - A Practical Introduction.* (CRC Press, 1996) [ISBN 978-0849380716]. An excellent though moderately difficult account of Itô integration, properties of Brownian motion, solution to stochastic differential equations.

Evans, L.C. *Partial Differential Equations.* (American Mathematical Society, Providence, 1998) [ISBN 978-0821807729].

Schuss, Z.*Theory and Applications of Stochastic Differential Equations*. (J. Wiley, 1980) [ISBN 978-0471043942]. A very down-to-earth text but directed at applications in physics. Discusses Stratonovich integration as well.

Merton, R.C. *Continuous-time Finance.* (Blackwell, 1996) [ISBN 978-0631185086].

Samuelson,P. 'Proof that properly anticipated prices fluctuate randomly', *Industrial Management Review,* 6(2) 1965, pp.41–49.

Further reading (historical, anecdotal, or popular texts)

Bernstein, P. *Against the Odds.* (J. Wiley, 1998) [ISBN 0471295639].

Bernstein, P. *Capital Ideas - The Improbable origins of modern Wall Street.* (J. Wiley, 1998) [ISBN 978-0029030127]. A wonderful history of the growth of ideas in this area - a very good read.

Davis, M. and A. Etheridge *Louis Bachelier's Theory of Speculation: the origins of modern finance.* Translated and with commentary (Princeton, 2006) [ISBN 978-0691117522].

Dunbar, N. *Inventing Money: The Story of Long-Term Capital Management and the Legends Behind It.* (J. Wiley, 2001) [ISBN 978-0471498117].

Kay, J. *Foundations of Corporate Success.* (Oxford University Press, 1995) [ISBN 978-0198289883].

Kay, J. *The truth about markets.* (Allen Lane, Penguin Press 2003) [ISBN 978-0140296723]. ('Everything you wanted to know about economics, but were afraid to ask' – Mervyn King, Governor, Bank of

England).

Lewis, M. *Liar's Poker.* (Coronet Books, 1989) [ISBN 978-0340767009].

Lowenstein, R. *When Genius Failed: The Rise and Fall of Long-Term Capital Management.* (Random House, 2000) [ISBN 978-0375758256].

Smithers, A. *Valuing Wall Street: Protecting Wealth in Turbulent Times.* (McGraw-Hill, 2002) [ISBN 978-0071354615 (hbk) 9780071387835 (pbk)].

Bachelier, L. *Louis Bachelier's Theory of Speculation: the origin of modern finance*. Translated and with commentary by Davis, M and A. Etheridge (Princeton: Princeton University Press, 2006) [ISBN 978-0691117522].

1.7 Using the subject guide

I have already mentioned that this guide is not a textbook. It is important that you read textbooks in conjunction with the guide and that you try problems from the textbooks. The learning activities throughout the guide, and the sample questions at the end of the chapters, are a very useful resource. You should try them once you think you have mastered a particular chapter. Do really try them: don't just simply read the solutions where provided. Make a serious attempt before consulting the solutions. Note that the solutions are often just sketch solutions, to indicate to you how to answer the questions, but in the examination, you must show **all** your calculations. It is vital that you develop and enhance your problem-solving skills and the only way to do this is to try lots of examples.

1.8 Examination advice

Important: Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current *Regulations* for relevant information about the examination, and the current *Examiners' reports* where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

A sample examination paper is given as an appendix to this guide. There are no optional topics in this syllabus: you should do them all.

Please be aware that a few sections in this guide have been starred (*) to indicate that the material of the section is not examinable. The material has been included in the guide either for its interest value or in order to help you to see the connection with a wider view of the subject.

Please do not assume that the questions in a real examination will necessarily be very similar to these sample questions. An examination is designed (by definition) to test you. You **will** get examination questions that are unlike any questions in this guide. The whole point of examining is to see whether you can apply knowledge in both familiar

and unfamiliar settings. The Examiners have an obligation to surprise you! For this reason, it is important that you try as many examples as possible: from the guide and from the textbooks. This is not so that you can cover any possible type of question the examiners can think of! It's so that you get used to confronting unfamiliar questions, grappling with them, and finally coming up with the solution.

Do not panic if you cannot completely solve an examination question. There are many marks to be awarded for using the correct approach or method.

A final point about the examination for this unit is to reassure you that examination questions will not assume familiarity with the contents of any of the starred sections in this subject guide. The purpose of such examples is to motivate the mathematics and to persuade you that the mathematics you are learning has important uses.

Acknowledgement

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Adam Ostaszewski