Mathematical economics
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Undergraduate study in
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This is an extract from a subject guide for an undergraduate course offered as part of the University of London International Programmes in Economics, Management, Finance and the Social Sciences. Materials for these programmes are developed by academics at the London School of Economics and Political Science (LSE).

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to pressure of work the authors are unable to enter into any correspondence relating to,
or arising from, the guide. If you have any comments on this subject guide, favourable or
unfavourable, please use the form at the back of this guide.
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Welcome to EC3120 Mathematical economics which is a 300 course offered on the Economics, Management, Finance and Social Sciences (EMFSS) suite of programmes. In this brief introduction, we describe the nature of this course and advise on how best to approach it. Essential textbooks and Further reading resources for the entire course are listed in this introduction for easy reference. At the end, we present relevant examination advice.

1.1 The structure of the course

The course consists of two parts which are roughly equal in length but belonging to the two different realms of economics:

- the first deals with the mathematical apparatus needed to rigorously formulate the core of microeconomics, the consumer choice theory
- the second presents a host of techniques used to model intertemporal decision making in the macroeconomy.

The two parts are also different in style. In the first part it is important to lay down rigorous proofs of the main theorems while paying attention to assumption details. The second part often dispenses with rigour in favour of slightly informal derivations needed to grasp the essence of the methods. The formal treatment of the underlying mathematical foundations is too difficult to be within the scope of undergraduate study; still, the methods can be used fruitfully without the technicalities involved: most macroeconomists actively employing them have never taken a formal course in optimal control theory.

As you should already have an understanding of multivariate calculus and integration, we have striven to make the exposition in both parts of the subject completely self-contained. This means that beyond the basic prerequisites you do not need to have an extensive background in fields like functional analysis, topology, or differential equations. However, such a background may allow you to progress faster. For instance, if you have studied the concepts and methods of ordinary differential equations before you may find that you can skip parts of Chapter 8.

By design the course has a significant economic component. Therefore we apply the techniques of constrained optimisation to the problems of static consumer and firm choice; the dynamic programming methods are employed to analyse consumption smoothing, habit formation and allocation of spending on durables and non-durables; the phase plane tools are used to study dynamic fiscal policy analysis and foreign
1. Introduction

currency reserves dynamics; Pontryagin’s maximum principle is utilised to examine a firm’s investment behaviour and the aggregate saving behaviour in an economy.

1.2 Aims

The course is specifically designed to:

- demonstrate to you the importance of the use of mathematical techniques in theoretical economics
- enable you to develop skills in mathematical modelling.

1.3 Learning outcomes

At the end of this course, and having completed the Essential reading and activities, you should be able to:

- use and explain the underlying principles, terminology, methods, techniques and conventions used in the subject
- solve economic problems using the mathematical methods described in the subject.

1.4 Syllabus

Techniques of constrained optimisation. This is a rigorous treatment of the mathematical techniques used for solving constrained optimisation problems, which are basic tools of economic modelling. Topics include: definitions of a feasible set and of a solution, sufficient conditions for the existence of a solution, maximum value function, shadow prices, Lagrangian and Kuhn-Tucker necessity and sufficiency theorems with applications in economics, for example General Equilibrium theory, Arrow-Debreau securities and arbitrage.

Intertemporal optimisation. Bellman approach. Euler equations. Stationary infinite horizon problems. Continuous time dynamic optimisation (optimal control). Applications, such as habit formation, Ramsay-Kass-Coopmans model, Tobin’s q, capital taxation in an open economy, are considered.

Tools for optimal control: ordinary differential equations. These are studied in detail and include linear second order equations, phase portraits, solving linear systems, steady states and their stability.

1.5 Reading advice

While topics covered in this subject are in every economists essential toolbox, their textbook coverage varies. There are a lot of first-rate treatments of static optimisation
1.5. Reading advice

methods; most textbooks that have ‘mathematical economics’ or ‘mathematics for economists’ in the title will have covered these in various levels of rigour. Therefore students with different backgrounds will be able to choose a book with the most suitable level of exposition.

1.5.1 Essential reading


The textbook by Dixit, is perhaps in the felicitous middle. However until recently there has been no textbook that covers all the aspects of the dynamic analysis and optimisation used in macroeconomic models.


The book by Sydsæter et al. is an attempt to close that gap, and is therefore the Essential reading for the second part of the course, despite the fact that the exposition is slightly more formal than in this guide. This book covers almost all of the topics in the course, although the emphasis falls on the technique and not on the proof. It also provides useful reference for linear algebra, calculus and basic topology. The style of the text is slightly more formal than the one adopted in this subject guide. For that reason we have included references to Further reading, especially from various macroeconomics textbooks, which may help develop a more intuitive (non-formal) understanding of the concepts from the application-centred perspective. Note that whatever your choice of Further reading textbooks is, textbook reading is essential. As with lectures, this guide gives structure to your study, while the additional reading supplies a lot of detail to supplement this structure. There are also more learning activities and Sample examination questions, with solutions, to work through in each chapter.

Detailed reading references in this subject guide refer to the editions of the set textbooks listed above. New editions of one or more of these textbooks may have been published by the time you study this course. You can use a more recent edition of any of the books; use the detailed chapter and section headings and the index to identify relevant readings. Also check the virtual learning environment (VLE) regularly for updated guidance on readings.

1.5.2 Further reading

Please note that as long as you read the Essential reading you are then free to read around the subject area in any text, paper or online resource. You will need to support your learning by reading as widely as possible and by thinking about how these principles apply in the real world. To help you read extensively, you have free access to the VLE and University of London Online Library (see below).

Other useful texts for this course include:


### 1.6 Online study resources

In addition to the subject guide and the reading, it is crucial that you take advantage of the study resources that are available online for this course, including the VLE and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at:

http://my.londoninternational.ac.uk

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You
have probably already logged in to the Student Portal in order to register! As soon as
you registered, you will automatically have been granted access to the VLE, Online
Library and your fully functional University of London email account.
If you have forgotten these login details, please click on the ‘ Forgotten your password’
link on the login page.

1.6.1 The VLE

The VLE, which complements this subject guide, has been designed to enhance your
learning experience, providing additional support and a sense of community. It forms an
important part of your study experience with the University of London and you should
access it regularly.
The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of
  subject material.
- Electronic study materials: The printed materials that you receive from the
  University of London are available to download, including updated reading lists
  and references.
- Past examination papers and Examiners’ commentaries: These provide advice on
  how each examination question might best be answered.
- A student discussion forum: This is an open space for you to discuss interests and
  experiences, seek support from your peers, work collaboratively to solve problems
  and discuss subject material.
- Videos: There are recorded academic introductions to the subject, interviews and
  debates and, for some courses, audio-visual tutorials and conclusions.
- Recorded lectures: For some courses, where appropriate, the sessions from previous
  years’ Study Weekends have been recorded and made available.
- Study skills: Expert advice on preparing for examinations and developing your
digital literacy skills.
- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our
provision all the time and you should check the VLE regularly for updates.

1.6.2 Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help
you read widely and extensively.
To access the majority of resources via the Online Library you will either need to use
your University of London Student Portal login details, or you will be required to
register and use an Athens login:
1. Introduction

http://tinyurl.com/ollathens
The easiest way to locate relevant content and journal articles in the Online Library is to use the Summon search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please see the online help pages:
www.external.shl.lon.ac.uk/summon/about.php

1.7 Using the subject guide

We have already mentioned that this guide is not a textbook. It is important that you read textbooks in conjunction with the subject guide and that you try problems from them. The Learning activities and the sample questions at the end of the chapters in this guide are a very useful resource. You should try them all once you think you have mastered a particular chapter. Do really try them: do not just simply read the solutions where provided. Make a serious attempt before consulting the solutions. Note that the solutions are often just sketch solutions, to indicate to you how to answer the questions but, in the examination, you must show all your calculations. It is vital that you develop and enhance your problem-solving skills and the only way to do this is to try lots of examples.

Finally, we often use the symbol ■ to denote the end of a proof, where we have finished explaining why a particular result is true. This is just to make it clear where the proof ends and the following text begins.

1.8 Examination

Important: Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current Regulations, for relevant information about the examination, and the VLE where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

A Sample examination paper is at the end of this guide. Notice that the actual questions may vary, covering the whole range of topics considered in the course syllabus. You are required to answer eight out of 10 questions: all five from Section A (8 marks each) and any three from Section B (20 marks each). Candidates are strongly advised to divide their time accordingly.

Also note that in the examination you should submit all your derivations and rough work. If you cannot completely solve an examination question you should still submit partial answers as many marks are awarded for using the correct approach or method.

Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
where available, past examination papers and *Examiners’ commentaries* for this course which give advice on how each question might best be answered.
1. Introduction
Chapter 2
Constrained optimisation: tools

2.1 Aim of the chapter

The aim of this chapter is to introduce you to the topic of constrained optimisation in a static context. Special emphasis is given to both the theoretical underpinnings and the application of the tools used in economic literature.

2.2 Learning outcomes

By the end of this chapter, you should be able to:

- formulate a constrained optimisation problem
- discern whether you could use the Lagrange method to solve the problem
- use the Lagrange method to solve the problem, when this is possible
- discern whether you could use the Kuhn-Tucker Theorem to solve the problem
- use the Kuhn-Tucker Theorem to solve the problem, when this is possible
- discuss the economic interpretation of the Lagrange multipliers
- carry out simple comparative statics using the Envelope Theorem.

2.3 Essential reading

This chapter is self-contained and therefore there is no essential reading assigned.

2.4 Further reading

- Sydsæter, K., P. Hammond, A. Seierstad and A. Strøm Further mathematics for economic analysis. Chapters 1 and 3.
- Dixit, A.K. Optimization in economics theory. Chapters 1–8 and the appendix.
2. Constrained optimisation: tools

2.5 Introduction

The role of optimisation in economic theory is important because we assume that individuals are rational. Why do we look at constrained optimisation? The problem of scarcity.

In this chapter we study the methods to solve and evaluate the constrained optimisation problem. We develop and discuss the intuition behind the Lagrange method and the Kuhn-Tucker theorem. We define and analyse the maximum value function, a construct used to evaluate the solutions to optimisation problems.

2.6 The constrained optimisation problem

Consider the following example of an economics application.

Example 2.1 ABC is a perfectly competitive, profit-maximising firm, producing \( y \) from input \( x \) according to the production function \( y = x^5 \). The price of output is 2, and the price of input is 1. Negative levels of \( x \) are impossible. Also, the firm cannot buy more than \( k \) units of input.

The firm is interested in two problems. First, the firm would like to know what to do in the short run; given that its capacity, \( k \), (e.g. the size of its manufacturing facility) is fixed, it has to decide how much to produce today, or equivalently, how many units of inputs to employ today to maximise its profits. To answer this question, the firm needs to have a method to solve for the optimal level of inputs under the above constraints.

When thinking about the long-run operation of the firm, the firm will consider a second problem. Suppose the firm could, at some cost, invest in increasing its capacity, \( k \), is it worthwhile for the firm? By how much should it increase/decrease \( k \)? To answer this the firm will need to be able to evaluate the benefits (in terms of added profits) that would result from an increase in \( k \).

Let us now write the problem of the firm formally:

\[
\begin{align*}
\text{(The firm’s problem)} & \quad \max g(x) = 2x^5 - x \\
\text{s.t.} & \quad h(x) = x \leq k \\
& \quad x \geq 0.
\end{align*}
\]

More generally, we will be interested in similar problems as that outlined above. Another important example of such a problem is that of a consumer maximising his utility trying to choose what to consume and constrained by his budget. We can formulate the general problem denoted by ‘COP’ as:

\[
\begin{align*}
\text{(COP)} & \quad \max g(x) \\
\text{s.t.} & \quad h(x) \leq k \\
& \quad x \in x.
\end{align*}
\]
Note that in the general formulation we can accommodate multidimensional variables. In particular $g : x \to \mathbb{R}$, $x$ is a subset of $\mathbb{R}^n$, $h : x \to \mathbb{R}^m$ and $k$ is a fixed vector in $\mathbb{R}^m$.

**Definition 1**

$z^*$ solves COP if $z^* \in Z$, $h(z^*) \leq k$, and for any other $z \in Z$ satisfying $h(z) \leq k$, we have that $g(z^*) \geq g(z)$.

**Definition 2**

The feasible set is the set of vectors in $\mathbb{R}^n$ satisfying $z \in Z$ and $h(z) \leq k$.

**Activity 2.1** Show that the following constrained maximisation problems have no solution. For each example write what you think is the problem for the existence of a solution.

(a) Maximise $\ln x$ subject to $x > 1$.
(b) Maximise $\ln x$ subject to $x < 1$.
(c) Maximise $\ln x$ subject to $x < 1$ and $x > 2$.

Solutions to activities are found at the end of the chapters.

To steer away from the above complications we can use the following theorem:

**Theorem 1**

If the feasible set is non-empty, closed and bounded (compact), and the objective function $g$ is continuous on the feasible set then the COP has a solution.

Note that for the feasible set to be compact it is enough to assume that $x$ is closed and the constraint function $h(x)$ is continuous. Why?

The conditions are sufficient and not necessary. For example $x^2$ has a minimum on $\mathbb{R}^2$ even if it is not bounded.

**Activity 2.2** For each of the following sets of constraints either say, without proof, what is the maximum of $x^2$ subject to the constraints, or explain why there is no maximum:

(a) $x \leq 0$, $x \geq 1$
(b) $x \leq 2$
(c) $0 \leq x \leq 1$
(d) $x \geq 2$
In what follows we will devote attention to two questions (similar to the short-run and long-run questions that the firm was interested in in the example). First, we will look for methods to solve the COP. Second, having found the optimal solution we would like to understand the relation between the constraint \( k \) and the optimal value of the COP given \( k \). To do this we will study the concept of the maximum value function.

2.7 Maximum value functions

To understand the answer to both questions above it is useful to follow the following route. Consider the example of firm ABC. A first approach to understanding the maximum profit that the firm might gain is to consider the situations that the firm is indeed constrained by \( k \). Plot the profit of the firm as a function of \( x \) and we see that profits are increasing up to \( x = 1 \) and then decrease, crossing zero profits when \( x = 4 \). This implies that the firm is indeed constrained by \( k \) when \( k \leq 1 \); in this case it is optimal for the firm to choose \( x^* = k \). But when \( k > 1 \) the firm is not constrained by \( k \); choosing \( x^* = 1 < k \) is optimal for the firm.

Since we are interested in the relation between the level of the constraint \( k \) and the maximum profit that the firm could guarantee, \( v \), it is useful to look at the plane spanned by taking \( k \) on the \( x \)-axis and \( v \) on the \( y \)-axis. For firm ABC it is easy to see that maximal profits follow exactly the increasing part of the graph of profits we had before (as \( x^* = k \)). But when \( k > 1 \), as we saw above, the firm will always choose \( x^* = 1 < k \) and so the maximum attainable profit will stay flat at a level of 1.

More generally, how can we think about the maximum attainable value for the COP? Formally, and without knowing if a solution exists or not, we can write this function as

\[
s(k) = \sup \{g(x) : x \in i, h(x) \leq k\}.
\]

We would like to get some insight as to what this function looks like. One way to proceed is to look, in the \((k, v)\) space, at all the possible points, \((k, v)\), that are attainable by the function \( g(x) \) and the constraints. Formally, consider the set,

\[
B = \{(k, v) : k \geq h(x), v \leq g(x) \text{ for some } x \in x\}.
\]

The set \( B \) defines the ‘possibility set’ of all the values that are feasible, given a constraint \( k \). To understand what is the maximum attainable value given a particular \( k \) is to look at the upper boundary of this set. Formally, one can show that the values \( v \) on the upper boundary of \( B \) correspond exactly with the function \( s(k) \), which brings us closer to the notion of the maximum value function.

It is intuitive that the function \( s(k) \) will be monotone in \( k \); after all, when \( k \) is increased, this cannot lower the maximal attainable value as we have just relaxed the
2.7. Maximum value functions

Figure 2.1: The set $B$ and $S(k)$.

constraints. In the example of firm $ABC$, abstracting away from the cost of the facility, a larger facility may never imply that the firm will make less profits! The following lemma formalises this.

**Lemma 1**

If $k_1 \in K$ and $k_1 \leq k_2$ then $k_2 \in K$ and $s(k_1) \leq s(k_2)$.

**Proof.** If $k_1 \in K$ there exists a $z \in Z$ such that $h(z) \leq k_1 \leq k_2$ so $k_2 \leq K$. Now consider any $v < s(k_1)$. From the definition there exists a $z \in Z$ such that $v < g(z)$ and $h(z) \leq k_1 \leq k_2$, which implies that $s(k_2) = \sup \{g(z); z \in Z, h(z) \leq k_2\} > v$. Since this is true for all $v < s(k_1)$ it implies that $s(k_2) \geq s(k_1)$. ■

Therefore, the boundary of $B$ defines a non-decreasing function. If the set $B$ is closed, that is, it includes its boundary, this is the maximum value function we are looking for. For each value of $k$ it shows the maximum available value of the objective function.

We can confine ourselves to maximum rather than supremum. This is possible if the COP has a solution. To ensure a solution, we can either find it, or show that the objective function is continuous and the feasible set is compact.

**Definition 3 (The maximum value function)**

If $z(k)$ solves the COP with the constraint parameter $k$, the maximum value function is $v(k) = g(z(k))$.

The maximum value function, it it exists, has all the properties of $s(k)$. In particular, it is non-decreasing. So we have reached the conclusion that $x^*$ is a solution to the COP if and only if $(k, g(x^*))$ lies on the upper boundary of $B$.

**Activity 2.3** $XYZ$ is a profit-maximising firm selling a good in a perfectly competitive market at price 4. It can produce any non-negative quantity of such good $y$ at cost $c(y) = y^2$. However there is a transport bottleneck which makes it
impossible for the firm to sell more than \( k \) units of \( y \), where \( k \geq 0 \). Write down \( XYZ \)'s profit maximisation problem. Show on a graph the set \( B \) for this position. Using the graph write down the solution to the problem for all non-negative values of \( k \).

### 2.8 The Lagrange sufficiency theorem

In the last section we defined what we mean by the maximum value function given that we have a solution. We also introduced the set, \( B \), of all feasible outcomes in the \((k,v)\) space. We concluded that \( x^\star \) is a solution to the COP if and only if \((k,g(x^\star))\) lies on the upper boundary of \( B \). In this section we proceed to use this result to find a method to solve the COP.

Which values of \( x \) would give rise to boundary points of \( B \)? Suppose we can draw a line with a slope \( \lambda \) through a point \((k,v) = (h(x^\star),g(x^\star))\) which lies entirely on or above the set \( B \). The equation for this line is:

\[
v - \lambda k = g(x^\star) - \lambda h(x^\star).
\]

**Example 2.2 (Example 2.1 revisited)** We can draw such a line through \((0.25,0.75)\) with slope \( q = 1 \) and a line through \((1,1)\) with slope \( 0 \).

If the slope \( \lambda \) is **non-negative** and recalling the definition of \( B \) as the set of all possible outcomes, the fact that the line lies entirely on or above the set \( B \) can be restated as

\[
g(x^\star) - \lambda h(x^\star) \geq g(x) - \lambda h(x)
\]

for all \( x \in x \). But note that this implies that \((h(x^\star),g(x^\star))\) lies on the upper boundary of the set \( B \) implying that if \( x^\star \) is feasible it is a solution to the COP.

**Example 2.3 (Example 2.1 revisited)** It is crucial for this argument that the slope of the line is non-negative. For example for the point \((4,0)\) there does not exist a line passing through it that is on or above \( B \). This corresponds to \( x = 4 \) and indeed it is not a solution to our example. Sometimes the line has a slope of \( q = 0 \). Suppose for example that \( k = 4 \), if we take the line with \( q = 0 \) through \((4,1)\), it indeed lies above the set \( B \). The point \( x^\star = 1 \) satisfies:

\[
g(x^\star) - 0h(x^\star) \geq g(x) - 0h(x)
\]

for all \( x \in x \), and as \( h(x^\star) < k \), then \( x^\star \) solves the optimisation problem for \( k = k^\star \geq 1 \).

Summarising the argument so far, suppose \( k^\star \), \( \lambda \) and \( x^\star \) satisfy the following conditions:

\[
g(x^\star) - \lambda h(x^\star) \geq g(x) - \lambda h(x) \quad \text{for all } x \in x
\]

\[
\lambda \geq 0
\]

\[
x^\star \in x
\]

either \( k^\star = h(x^\star) \)

or \( k^\star > h(x^\star) \) and \( \lambda = 0 \)
then \( x^* \) solves the COP for \( k = k^* \).

This is the Lagrange sufficiency theorem. It is convenient to write it slightly differently: adding \( \lambda k^* \) to both sides of the first condition we have

\[
\begin{align*}
g(x^*) + \lambda (k^* - h(x^*)) & \geq g(x) + \lambda (k^* - h(x)) \quad \text{for all } x \in x \setminus 0 \\
\lambda & \geq 0 \\
x^* & \in x \\
k^* & \geq h(x^*) \\
\lambda[k^* - h(x^*)] & = 0.
\end{align*}
\]

We refer to \( \lambda \) as the Lagrange multiplier. We refer to the expression \( g(x^*) + \lambda (k^* - h(x^*)) \equiv \mathcal{L}(x, k^*, \lambda) \) as the Lagrangian. The conditions above imply that \( x^* \) maximises the Lagrangian given a non-negativity restriction, feasibility, and a complementary slackness \((\text{CS})\) condition respectively. Formally:

**Theorem 2**

If for some \( q \geq 0 \), \( z^* \) maximises \( \mathcal{L}(z, k^*, q) \) subject to the three conditions, it also solves the COP.

**Proof.** From the complementary slackness condition, \( q[k^* - h(z^*)] = 0 \). Thus, \( g(z^*) = g(z^*) + q(k^* - h(z^*)) \). By \( q \geq 0 \) and \( k^* - h(z) \geq 0 \) for all feasible \( z \), then \( g(z) + q(k^* - h(z)) \geq g(z) \). By maximisation of \( \mathcal{L} \) we get \( g(z^*) \geq g(z) \), for all feasible \( z \) and since \( z^* \) itself is feasible, then it solves the COP. \( \blacksquare \)

**Example 2.4** (Example 2.1 revisited) We now solve the example of the firm ABC. The Lagrangian is given by

\[
\mathcal{L}(x, k, \lambda) = 2x^{0.5} - x + \lambda(k - z).
\]

Let us use first order conditions, although we have to prove that we can use them and we will do so later in the chapter. Given \( \lambda \), the first order condition of \( \mathcal{L}(x, k^*, \lambda) \) with respect to \( x \) is

\[
x^{-0.5} - 1 - \lambda = 0 \quad (\text{FOC}).
\]

We need to consider the two cases (using the CS condition), \( \lambda > 0 \) and the case of \( \lambda = 0 \). Case 1: If \( \lambda = 0 \). CS implies that the constraint is not binding and (FOC) implies that \( x^* = 1 \) is a candidate solution. By Theorem 1, when \( x^* \) is feasible, i.e. \( k \geq x^* = 1 \), this will indeed be a solution to the COP. Case 2: If \( \lambda > 0 \). In this case the constraint is binding; by CS we have \( x^* = k \) as a candidate solution. To check that it is a solution we need to check that it is feasible, i.e that \( k > 0 \), that it satisfies the FOC and that it is consistent with a non-negative \( \lambda \). From the FOC we have

\[
k^{-0.5} - 1 = \lambda
\]

and for this to be non-negative implies that:

\[
k^{-0.5} - 1 \geq 0 \iff k \leq 1.
\]

Therefore when \( k \leq 1 \), \( x^* = k \) solves the COP.
2. Constrained optimisation: tools

Figure 2.2: Note that the point \( C \) represents a solution to the COP when \( k = k^* \), but this point will not be characterised by the Lagrange method as the set \( B \) is not concave.

Remark 1 Some further notes about what is to come:

- The conditions that we have stated are sufficient conditions. This means that some solutions of COP cannot be characterised by the Lagrangian. For example, if the set \( B \) is not convex, then solving the Lagrangian is not necessary.

- As we will show later, if the objective function is concave and the constraint is convex, then \( B \) is convex. Then, the Lagrange conditions are also necessary. That is, if we find all the points that maximise the Lagrangian, these are all the points that solve the COP.

- With differentiability, we can also solve for these points using first order conditions.

- Remember that we are also interested in the maximum value function. What does it mean to relax the constraint? The Lagrange multipliers are going to play a central role in understanding how relaxing the constraints affect the maximum value.

2.9 Concavity and convexity and the Lagrange Necessity Theorem

In the last section we have found necessary conditions for a solution to the COP. This means that some solutions of COP cannot be characterised by the Lagrangian method. In this section we investigate the assumption that would guarantee that the conditions of the Lagrangian are also necessary.

To this end, we will need to introduce the notions of convexity and concavity. From the example above it is already clear why convexity should play a role: if the set \( B \) is not convex, there will be points on the boundary of \( B \) (that as we know are solutions to the COP) that will not accommodate a line passing through them and entirely above the set \( B \) as we did in the last section. In turn, this means that using the Lagrange method will not lead us to these points.
2.9. Concavity and convexity and the Lagrange Necessity Theorem

We start with some formal definitions:

**Definition 4**

A set $U$ is a convex set if for all $x \in U$ and $y \in U$, then for all $t \in [0, 1]$:  
$$tx + (1-t)y \in U.$$

**Definition 5**

A real-valued function $f$ defined on a convex subset of $U$ of $\mathbb{R}^n$ is concave, if for all $x, y \in U$ and for all $t \in [0, 1]$:  
$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

A real-valued function $g$ defined on a convex subset $U$ of $\mathbb{R}^n$ is convex, if for all $x, y \in U$ and for all $t \in [0, 1]$:  
$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y).$$

**Remark 2** Some simple implications of the above definition that will be useful later:

- $f$ is concave if and only if $-f$ is convex.
- Linear functions are convex and concave.
- Concave and convex functions need to have convex sets as their domain. Otherwise, we cannot use the conditions above.

**Activity 2.4** $A$ and $B$ are two convex subsets of $\mathbb{R}^n$. Which of the following sets are always convex, sometimes convex, or never convex? Provide proofs for the sets which are always convex, draw examples to show why the others are sometimes or never convex.

(a) $A \cup B$

(b) $A + B \equiv \{x \mid x \in \mathbb{R}^n, x = a + b, a \in A, b \in B\}$.

In what follows we will need to have a method of determining whether a function is convex, concave or neither. To this end the following characterisation of concave functions is useful:
2. Constrained optimisation: tools

Let $f$ be a continuous and differentiable function on a convex subset $U$ of $\mathbb{R}^n$. Then $f$ is concave on $U$ if and only if for all $x,y \in U$:

$$f(y) - f(x) \leq Df(x)(y - x) = \frac{\partial f(x)}{\partial x_1}(y_1 - x_1) + \cdots + \frac{\partial f(x)}{\partial x_n}(y_n - x_n).$$

**Lemma 2**

**Proof.** Here we prove the result on $\mathbb{R}^1$: since $f$ is concave, then:

$$tf(y) + (1 - t)f(x) \leq f(ty + (1 - t)x) \Leftrightarrow$$

$$t(f(y) - f(x)) + f(x) \leq f(x + t(y - x)) \Leftrightarrow$$

$$f(y) - f(x) \leq \frac{f(x + t(y - x)) - f(x)}{t} \Leftrightarrow$$

$$f(y) - f(x) \leq \frac{f(x + h) - f(x)}{h}(y - x)$$

for $h = t(y - x)$. Taking limits when $h \to 0$ this becomes:

$$f(y) - f(x) \leq f'(x)(y - x).$$

■

Remember that we introduced the concepts of concavity and convexity as we were interested in finding out under what conditions is the Lagrange method also a necessary condition for solutions of the COP.

Consider the following assumptions, denoted by CC:

1. The set $x$ is convex.
2. The function $g$ is concave.
3. The function $h$ is convex.

To see the importance of these assumptions, recall the definition of the set $B$:

$$B = \{(k, v) : k \geq h(x), v \leq g(x) \text{ for some } x \in x\}.$$ 

**Proposition 1**

Under assumptions CC, the set $B$ is convex.

**Proof.** Suppose that $(k_1, v_1)$ and $(k_2, v_2)$ are in $B$, so there exists $z_1$ and $z_2$ such that:

$$k_1 \geq h(z_1)k_2 \geq h(z_2)$$

$$v_1 \leq g(z_1)v_2 \geq g(z_2).$$

By convexity of $h$:

$$\theta k_1 + (1 - \theta)k_2 \geq \theta h(z_1) + (1 - \theta)h(z_2) \geq h(\theta z_1 + (1 - \theta)z_2)$$
and by concavity of $g$:

$$\theta v_1 + (1 - \theta)v_2 \leq \theta g(z_1) + (1 - \theta)g(z_2) \leq g(\theta z_1 + (1 - \theta)z_2)$$

thus, $(\theta k_1 + (1 - \theta)k_2, \theta v_1 + (1 - \theta)v_2) \in B$ for all $\theta \in [0, 1]$, implying that $B$ is convex. ■

Remember that the maximum value is the upper boundary of the set $B$. When $B$ is convex, we can say something about the shape of the maximum value function:

**Proposition 2**

Assume that the maximum value exists, then under CC, the maximum value is a non-decreasing, concave and continuous function of $k$.

**Proof.** We have already shown, without assuming convexity or concavity, that the maximum value is non-decreasing. We have also shown that if the maximum value function $v(k)$ exists, it is the upper boundary of the set $B$. Above we proved that under CC the set $B$ is convex. The set $B$ can be re-written as:

$$B = \{(k, v) : v \in \mathbb{R}, k \in K, v \leq v(k)\}.$$ 

But a set $B$ is convex iff the function $v$ is concave. Thus, $v$ is concave, and concave functions are continuous, so $v$ is continuous. ■

### 2.10 The Lagrangian necessity theorem

We are now ready to formalise under what conditions the Lagrange method is necessary for a solution to the COP.

**Theorem 3**

Assume CC. Assume that the constraint qualification holds, that is, there is a vector $z_0 \in Z$ such that $h(z_0) \ll k^*$. Finally suppose that $z^*$ solves COP. Then:

i. there is a vector $q \in \mathbb{R}^m$ such that $z^*$ maximises the Lagrangian $\mathcal{L}(q, k^*, z) = g(z) + q[k^* - h(z^*)]$.

ii. the Lagrange multiplier $q$ is non-negative for all components, $q \geq 0$.

iii. the vector $z^*$ is feasible, that is $z \in Z$ and $h(z^*) \leq k$.

iv. the complementary slackness conditions are satisfied, that is, $q[k^* - h(z^*)] = 0$.

### 2.11 First order conditions: when can we use them?

So far we have found a method that allows us to find all the solutions to the COP by solving a modified maximisation problem (i.e. maximising the Lagrangian). As you recall, we have used this method to solve for our example of firm ABC by looking at
first order conditions. In this section we ask under what assumptions can we do this and be sure that we have found all the solutions to the problem. For this we need to introduce ourselves to the notions of continuity and differentiability.

### 2.11.1 Necessity of first order conditions

We start with some general definitions.

**Definition 6**

A function \( g : Z \to \mathbb{R}^m, Z \subset \mathbb{R}^n \), is differentiable at a point \( z_0 \) in the interior of \( Z \) if there exists a unique \( m \times n \) matrix, \( Dg(z_0) \), such that given any \( \epsilon > 0 \), there exists a \( \delta > 0 \), such that if \( |z - z_0| < \delta \), then \( |g(z) - g(z_0) - Dg(z_0)(z - z_0)| < \epsilon |z - z_0| \).

There are a few things to note about the above definition. First, when \( m = n = 1 \), \( Dg(x_0) \) is a scalar, that is, the derivative of the function at \( x_0 \) that we sometimes denote by \( g'(x_0) \). Second, one interpretation of the derivative is that it helps approximate the function \( g(x) \) for \( x \)'s that are close to \( x_0 \) by looking at the line, \( g(x_0) + Dg(x_0)(x - x_0) \), that passes through \( (x_0, g(x_0)) \) with slope \( Dg(x_0) \). Indeed this is an implication of Taylor’s theorem which states that for any \( n > 1 \):

\[
g(x) - g(x_0) = Dg(x_0)(x - x_0) + \frac{D^2g(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{D^ng(x_0)}{n!}(x - x_0)^n + R_n \tag{2.1}
\]

where \( R_n \) is a term of order of magnitude \( (x - x_0)^{n+1} \). The implication is that as we get closer to \( x_0 \) we can more or less ignore the elements with \( (x - x_0)^k \) with the highest powers and use just the first term to approximate the change in the function. We will later return to this when we will ask whether first order conditions are sufficient, but for now we focus on whether they are necessary.

But (2.1) implies that if \( x_0 \) is a point which maximises \( g(x) \) and is interior to the set we are maximising over then if \( Dg(x_0) \) exists then it must equal zero. If this is not the case, then there will be a direction along which the function will increase (i.e., the left-hand side of (2.1) will be positive); this is easily seen if we consider a function on one variable, but the same intuition generalises. This leads us to the following result:

**Theorem 4 (Necessity)**

Suppose that \( g : Z \to \mathbb{R}, Z \subset \mathbb{R}^n \), is differentiable in \( z_0 \) and that \( z_0 \in Z \) maximises \( g \) on \( Z \), then \( Dg(z_0) = 0 \).

Finally, we introduce the notion of continuity:

**Definition 7**

\( f : \mathbb{R}^k \to \mathbb{R}^m \) is continuous at \( x_0 \in \mathbb{R}^k \) if for any sequence \( \{x_n\}_{n=1}^\infty \in \mathbb{R}^k \) which converges to \( x_0 \), \( \{f(x_n)\}_{n=1}^\infty \) converges to \( f(x_0) \). The function \( f \) is continuous if it is continuous at any point in \( \mathbb{R}^k \).
Note that all differentiable functions are continuous but the converse is not true.

### 2.11.2 Sufficiency of first order conditions

Recall that if $f$ is a continuous and differentiable concave function on a convex set $U$ then

$$f(y) - f(x) \leq Df(x)(y - x).$$

Therefore, if we know that for some $x_0, y \in U$,

$$Df(x_0)(y - x_0) \leq 0$$

we have

$$f(y) - f(x_0) \leq Df(x_0)(y - x_0) \leq 0$$

implying that

$$f(y) \leq f(x_0).$$

If this holds for all $y \in U$, then $x_0$ is a global maximiser of $f$. This leads us to the following result:

#### Proposition 3 (Sufficiency)

Let $f$ be a continuous twice differentiable function whose domain is a convex open subset $U$ of $\mathbb{R}^n$. If $f$ is a concave function on $U$ and $Df(x_0) = 0$ for some $x_0$, then $x_0$ is a global maximum of $f$ on $U$.

Another way to see this result is to reconsider the Taylor expansion outlined above. Assume for the moment that a function $g$ is defined on one variable $x$. Remember that for $n = 2$, we have

$$g(x) - g(x_0) = Dg(x_0)(x - x_0) + \frac{D^2g(x_0)}{2!}(x - x_0)^2 + R_3. \quad (2.2)$$

If the first order conditions hold at $x_0$ this implies that $Dg(x_0) = 0$ and the above expression can be rewritten as,

$$g(x) - g(x_0) = \frac{D^2g(x_0)}{2!}(x - x_0)^2 + R_3. \quad (2.3)$$

But now we can see that when $D^2g(x_0) < 0$ and when we are close to $x_0$ the left-hand side will be negative and so $x_0$ is a local maximum of $g(x)$, and when $D^2g(x_0) > 0$ similarly $x_0$ will constitute a local minimum of $g(x)$. As concave functions have $D^2g(x) < 0$ for any $x$ and convex functions have $D^2g(x) > 0$ for any $x$ this shows why the above result holds.

The above intuition was provided for the case of a function over one variable $x$. In the next section we extend this intuition to functions on $\mathbb{R}^n$ to discuss how to characterise convexity and concavity in general.
2.11.3 Checking for concavity and convexity

In the last few sections we have introduced necessary and sufficient conditions for using first order conditions to solve maximisation problems. We now ask a more practical question. Confronted with a particular function, how can we verify whether it is concave or not? If it is concave, we know from the above results that we can use the first order conditions to characterise all the solutions. If it is not concave, we will have to use other means if we want to characterise all the solutions.

It will be again instructive to look at the Taylor expansion for \( n = 2 \). Let us now consider a general function defined on \( \mathbb{R}^n \) and look at the Taylor expansion around a vector \( x^0 \in \mathbb{R}^n \). As \( g \) is a function defined over \( \mathbb{R}^n \), \( Dg(x_0) \) is now an \( n \times n \) matrix. Let \( x \) be an \( n \)-dimensional vector in \( \mathbb{R}^n \). The Taylor expansion in this case becomes

\[
 g(x) - g(x^0) = Dg(x_0)(x - x^0) + (x - x^0)^T \frac{D^2 g(x_0)}{2!}(x - x^0) + R_3.
\]

If the first order condition is satisfied, then \( Dg(x_0) = 0 \), where \( 0 \) is the \( n \)-dimensional vector of zeros. This implies that we can write the above as

\[
 g(x) - g(x^0) = (x - x^0)^T \frac{D^2 g(x_0)}{2!}(x - x^0) + R_3.
\]

But now our problem is a bit more complicated. We need to determine the sign of \( (x - x^0)^T \frac{D^2 g(x_0)}{2!}(x - x^0) \) for a whole neighbourhood of \( x \)s around \( x^0 \). We need to find what properties of the matrix \( D^2 g(x_0) \) would guarantee this. For this we analyse the properties of expressions of the form \( (x - x^0)^T \frac{D^2 g(x_0)}{2!}(x - x^0) \), i.e. quadratic forms.

Consider functions of the form \( Q(x) = x^T A x \) where \( x \) is an \( n \)-dimensional vector and \( A \) a symmetric \( n \times n \) matrix. If \( n = 2 \), this becomes

\[
 \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

and can be rewritten as

\[
 a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2.
\]

**Definition 8**

i. A quadratic form on \( \mathbb{R}^n \) is a real-valued function

\[
 Q(x_1, x_2, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j
\]

or equivalently:

ii. A quadratic form on \( \mathbb{R}^n \) is a real-valued function \( Q(x) = x^T A x \) where \( A \) is a symmetric \( n \times n \) matrix.

Below we would like to understand what properties of \( A \) relate to the quadratic form it generates, taking on only positive values or only negative values.
2.11.4 Definiteness of quadratic forms

We now examine quadratic forms, \( Q(x) = x^T A x \). It is apparent that whenever \( x = 0 \) this expression is equal to zero. In this section we ask under what conditions \( Q(x) \) takes on a particular sign for any \( x \neq 0 \) (strictly negative or positive, non-negative or non-positive).

For example, in one dimension, when
\[
y = ax^2
\]
then if \( a > 0 \), \( ax^2 \) is non-negative and equals 0 only when \( x = 0 \). This is positive definite. If \( a < 0 \), then the function is negative definite. In two dimensions,
\[
x_1^2 + x_2^2
\]
is positive definite, whereas
\[
-x_1^2 - x_2^2
\]
is negative definite, whereas
\[
x_1^2 - x_2^2
\]
is indefinite, since it can take both positive and negative values, depending on \( x \).

There could be two intermediate cases: if the quadratic form is always non-negative but also equals 0 for non-zero \( x \)s, then we say it is positive semi-definite. This is the case, for example, for
\[
(x_1 + x_2)^2
\]
which can be 0 for points such that \( x_1 = -x_2 \). A quadratic form which is never positive but can be zero at points other than the origin is called negative semi-definite.

We apply the same terminology for the symmetric matrix \( A \), that is, the matrix \( A \) is positive semi-definite if \( Q(x) = x^T A x \) is positive semi-definite, and so on.

**Definition 9**

Let \( A \) be an \( n \times n \) symmetric matrix. Then \( A \) is:

- positive definite if \( x^T A x > 0 \) for all \( x \neq 0 \in \mathbb{R}^n \)
- positive semi-definite if \( x^T A x \geq 0 \) for all \( x \neq 0 \in \mathbb{R}^n \)
- negative definite if \( x^T A x < 0 \) for all \( x \neq 0 \in \mathbb{R}^n \)
- negative semi-definite if \( x^T A x \leq 0 \) for all \( x \neq 0 \in \mathbb{R}^n \)
- indefinite if \( x^T A x > 0 \) for some \( x \neq 0 \in \mathbb{R}^n \) and \( x^T A x < 0 \) for some \( x \neq 0 \in \mathbb{R}^n \).

2.11.5 Testing the definiteness of a matrix

In this section, we try to examine what properties of the matrix, \( A \), of the quadratic form \( Q(x) \) will determine its definiteness.
2. Constrained optimisation: tools

We start by introducing the notion of a **determinant** of a matrix. The determinant of a matrix is a unique scalar associated with the matrix.

Computing the determinant of a matrix proceeds recursively.

For a $2 \times 2$ matrix, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the determinant, denoted by $\det(A)$ or $|A|$, is:

$$a_{11}a_{22} - a_{12}a_{21}.$$

For a $3 \times 3$ matrix, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ the determinant is:

$$a_{11}\det\left(\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\right) - a_{12}\det\left(\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}\right) + a_{13}\det\left(\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\right).$$

Generally, for an $n \times n$ matrix, $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, the determinant will be given by:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i-1}a_{1i}\det(A_{1i})$$

where $A_{1i}$ is the matrix that is left when we take out the first row and $i$th column of the matrix $A$.

**Definition 10**

Let $A$ be an $n \times n$ matrix. A $k \times k$ submatrix of $A$ formed by deleting $n-k$ columns, say columns $i_1, i_2, \ldots, i_{n-k}$ and the same $n-k$ rows from $A$, is called a $k$th order principal submatrix of $A$. The determinant of a $k \times k$ principal submatrix is called a $k$th order principal minor of $A$.

**Example 2.5** For a general $3 \times 3$ matrix $A$, there is one third order principal minor, which is $\det(A)$. There are three second order principal minors and three first order principal minors. What are they?

**Definition 11**

Let $A$ be an $n \times n$ matrix. The $k$-th order principal submatrix of $A$ obtained by deleting the last $n-k$ rows and columns from $A$ is called the $k$-th order leading principal submatrix of $A$, denoted by $A_k$. Its determinant is called the $k$-th order leading principal minor of $A$, denoted by $|A_k|$.

We are now ready to relate the above elements of the matrix $A$ to the definiteness of the matrix:
2.11. First order conditions: when can we use them?

Let $A$ be an $n \times n$ symmetric matrix. Then:

(a) $A$ is positive definite if and only if all its $n$ leading principal minors are strictly positive.

(b) $A$ is negative definite if and only if all its $n$ leading principal minors alternate in sign as follows:

$$A_1 < 0, \ A_2 > 0, \ A_3 < 0, \ etc.$$ 

The $k$-th order leading principal minor should have the same sign as $(-1)^k$.

(c) $A$ is positive semi-definite if and only if every principal minor of $A$ is non-negative.

(d) $A$ is negative semi-definite if and only if every principal minor of odd order is non-positive and every principal minor of even order is non-negative.

**Proposition 4**

Example 2.6 Consider diagonal matrices:

$$
egin{pmatrix}
  a_1 & 0 & 0 \\
  0 & a_2 & 0 \\
  0 & 0 & a_3
\end{pmatrix}
$$

These correspond to the simplest quadratic forms:

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2.$$ 

This quadratic form will be positive (negative) definite if and only if all the $a_i$s are positive (negative). It will be positive semi-definite if and only if all the $a_i$s are non-negative and negative semi-definite if and only if all the $a_i$s are non-positive. If there are two $a_i$s of opposite signs, it will be indefinite. How do these conditions relate to what you get from the proposition above?

Example 2.7 To see how the conditions of the Proposition 4 relate to the definiteness of a matrix consider a $2 \times 2$ matrix, and in particular its quadratic form:

$$Q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2.$$ 

If $a = 0$, then $Q$ cannot be negative or positive definite since $Q(1, 0) = 0$. So assume that $a \neq 0$ and add and subtract $b^2x_2^2/a$ to get:

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 + \frac{b^2}{a} \frac{x_2^2}{a} - \frac{b^2}{a} \frac{x_2^2}{a}$$ 

$$= a \left( x - \frac{b^2}{a} \frac{x_2^2}{a} \right) + \frac{(ac - b^2)}{a} x_2^2.$$ 

$$= a \left( \frac{x_1 + b}{a} x_2^2 \right)^2 + \frac{(ac - b^2)}{a} x_2^2.$$ 


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If both coefficients above, $a$ and $(ac - b^2)/a$ are positive, then $Q$ will never be negative. It will equal 0 only when $x_1 + (b/a)x_2$ and $x_2 = 0$ in other words, when $x_1 = 0$ and $x_2 = 0$. Therefore, if:

$$|a| > 0 \quad \text{and} \quad \det A = \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$$

then $Q$ is positive definite. Conversely, in order for $Q$ to be positive definite, we need both $a$ and $\det A = ac - b^2$ to be positive. Similarly, $Q$ will be negative definite if and only if both coefficients are negative, which occurs if and only if $a < 0$ and $ac - b^2 > 0$, that is, when the leading principal minors alternate in sign. If $ac - b^2 < 0$, then the two coefficients will have opposite signs and $Q$ will be indefinite.

Example 2.8  Numerical examples. Consider $A = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$. Since $|A_1| = 2$ and $|A_2| = 5$, $A$ is positive definite. Consider $B = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$. Since $|B_1| = 2$ and $|B_2| = -2$, $B$ is indefinite.

2.11.6  Back to concavity and convexity

Finally we can put all the ingredients together. A continuous twice differentiable function $f$ on an open convex subset $U$ of $\mathbb{R}^n$ is concave on $U$ if and only if the Hessian $D^2 f(x)$ is negative semi-definite for all $x$ in $U$. The function $f$ is a convex function if and only if $D^2 f(x)$ is positive semi-definite for all $x$ in $U$.

Therefore, we have the following result:

**Proposition 5**

**Second order sufficient conditions for global maximum (minimum) in $\mathbb{R}^n$**

Suppose that $x^*$ is a critical point of a function $f(x)$ with continuous first and second order partial derivatives on $\mathbb{R}^n$. Then $x^*$ is:

- a global maximiser for $f(x)$ if $D^2 f(x)$ is negative (positive) semi-definite on $\mathbb{R}^n$.
- a strict global maximiser for $f(x)$ if $D^2 f(x)$ is negative (positive) definite on $\mathbb{R}^n$.

The property that critical points of concave functions are global maximisers is an important one in economic theory. For example, many economic principles, such as marginal rate of substitution equals the price ratio, or marginal revenue equals marginal cost are simply the first order necessary conditions of the corresponding maximisation problem as we will see. Ideally, an economist would like such a rule also to be a sufficient condition guaranteeing that utility or profit is being maximised, so it can provide a guideline for economic behaviour. This situation does indeed occur when the objective function is concave.


2.12 The Kuhn-Tucker Theorem

We are now in a position to formalise the necessary and sufficient first order conditions for solutions to the COP. Consider once again the COP:

\[
\begin{align*}
\text{max } & g(x) \\
\text{s.t. } & h(x) \leq k^* \\
& x \in x.
\end{align*}
\]

where \(g : x \to R, x\) is a subset of \(\mathbb{R}^n\), \(h : x \to \mathbb{R}^m\), and \(k^*\) is a fixed \(m\)-dimensional vector.

We impose a set of the following assumptions, CC’:

1. The set \(x\) is convex.
2. The function \(g\) is concave.
3. The function \(h\) is convex (these are assumptions CC from before), and
4. The functions \(g\) and \(h\) are differentiable.

Consider now the following conditions, which we term the Kuhn-Tucker conditions:

1. There is a vector \(\lambda \in \mathbb{R}^m\) such that the partial derivative of the Lagrangian:

\[
\mathcal{L}(k^*, \lambda, x) = g(x) + \lambda[k^* - h(x)]
\]

evaluated at \(x^*\) is zero, in other words:

\[
\frac{\partial \mathcal{L}(k^*, \lambda, x)}{\partial x} = Dg(x^*) - \lambda Dh(x^*) = 0.
\]

2. The Lagrange multiplier vector is non-negative:

\[
\lambda \geq 0.
\]

3. The vector \(x^*\) is feasible, that is, \(x \in x\) and \(h(x^*) \leq k^*\).

4. The complementary slackness conditions are satisfied, that is,

\[
\lambda[k^* - h(x^*)] = 0.
\]

The following theorem is known as the Kuhn-Tucker (K-T) Theorem.

**Theorem 5**

Assume CC’.

i. If \(z^*\) is in the interior of \(Z\) and satisfies the K-T conditions, then \(z^*\) solves the COP.

ii. If the constraint qualification holds (there exists a vector \(z_0 \in Z\) such that \(h(z_0) \ll k^*\)), \(z^*\) is in the interior of \(Z\) and solves the COP, then there is a vector of Lagrange multipliers \(q\) such that \(z^*\) and \(q\) satisfy the K-T conditions.
2. Constrained optimisation: tools

**Proof.** We first demonstrate that under CC and for non-negative values of Lagrange multipliers, the Lagrangian is concave. The Lagrangian is:

\[ \mathcal{L}(k^*, q, z) = g(z) + q[k^* - h(z)]. \]

Take \( z \) and \( z' \). Then:

\[
\begin{align*}
tg(z) + (1 - t)g(z') &\leq g(tz + (1 - t)z') \\
\frac{h(z)}{1-t} + (1-t)h(z') &\geq h(tz + (1 - t)z')
\end{align*}
\]

and thus with \( q \geq 0 \), we have:

\[
\begin{align*}
g(tz + (1 - t)z') + qk^* - qh(tz + (1 - t)z') &\geq tg(z) + (1 - t)g(z') + qk^* - q(th(z) + (1 - t)h(z')).
\end{align*}
\]

It follows that:

\[
\begin{align*}
\mathcal{L}(k^*, q, tz + (1 - t)z) &= g(tz + (1 - t)z') + q[k^* - h(tz + (1 - t)z')] \\
&\geq t[g(z) + q(k^* - h(z))] + (1 - t)[g(z') + q(k^* - h(z'))] \\
&= tL(k^*, q, z) + (1 - t)L(k^*, q, z').
\end{align*}
\]

This proves that the Lagrangian is concave in \( z \). In addition, we know that \( g \) and \( h \) are differentiable, therefore also \( \mathcal{L} \) is a differentiable function of \( z \). Thus, we know that if the partial derivative of \( \mathcal{L} \) with respect to \( z \) is zero at \( z^* \), then \( z^* \) maximises \( \mathcal{L} \) on \( Z \).

Indeed the partial derivative of \( L \) at \( z^* \) is zero, and hence we know that if \( g \) is concave and differentiable, \( h \) is convex and differentiable, the Lagrange multipliers \( q \) are non-negative, and

\[ Dg(z^*) - q Dh(z^*) = 0 \]

then \( z^* \) maximises the Lagrangian on \( Z \). But then the conditions of the Lagrange sufficiency theorem are satisfied, so that \( z^* \) indeed solves the COP. We have to prove the converse result now. Suppose that the constraint qualification is satisfied. The COP now satisfies all the conditions of the Lagrange necessity theorem. This theorem says that if \( z^* \) solves the COP, then it also maximises \( \mathcal{L} \) on \( Z \), and satisfies the complementary slackness conditions, with non-negative Lagrange multipliers, as well as being feasible. But since partial derivatives of a differentiable function are zero at the maximum, then the partial derivatives of \( \mathcal{L} \) with respect to \( z \) at \( z^* \) are zero and therefore all the Kuhn-Tucker conditions are satisfied. ■

**Remark 3 (Geometrical intuition)** Think of the following example in \( \mathbb{R}^2 \). Suppose that the constraints are

1. \( h_1(z) = -z_1 \leq 0. \)
2. \( h_2(z) = -z_2 \leq 0. \)
3. \( h_3(z) \leq k^*. \)

Consider first the case in which the point \( z_0 \) solves the problem at a tangency of \( h_3(z) = k^* \) and the objective function \( g(z) = g(z_0) \). The constraint set is convex, and by the concavity of \( g \) it is also the case that:

\[ \{ z : z \in \mathbb{R}^2, \ g(z) \geq g(z_0) \} \]
is a convex set. The Lagrangian for the problem is:

\[ L(k^*, q, z) = g(z) + q[k^* - h_3(z)] \]
\[ = g(z) + q_1 z_1 + q_2 z_2 + q_3 (k^* - h_3(z)). \]

In the first case of \( z_0 \), the non-negativity constraints do not bind. By the complementary slackness then, it is the case that \( q_1 = q_2 = 0 \) so that the first order condition is simply:

\[ Dg(z_0) = q_3 Dh_3(z_0). \]

Recall that \( q_3 \geq 0 \). If \( q_3 = 0 \), then it implies that \( Dg(z_0) = 0 \) so \( z_0 \) is the unconstrained maximiser but this is not the case here. Then \( q_3 > 0 \), which implies that the vectors \( Dg(z_0) \) and \( Dh_3(z_0) \) point in the same direction. These are the gradients: they describe the direction in which the function increases most rapidly. In fact, they must point in the same direction, otherwise, this is not a solution to the optimisation problem.

### 2.13 The Lagrange multipliers and the Envelope Theorem

#### 2.13.1 Maximum value functions

In this section we return to our initial interest in maximum (minimum) value functions. Profit functions and indirect utility functions are notable examples of maximum value functions, whereas cost functions and expenditure functions are minimum value functions. Formally, a maximum value function is defined by:

**Definition 12**

If \( x(b) \) solves the problem of maximising \( f(x) \) subject to \( g(x) \leq b \), the maximum value function is \( v(b) = f(x(b)) \).

You will remember that such a maximum value function is non-decreasing.

Let us now examine these functions more carefully. Consider the problem of maximising \( f(x_1, x_2, \ldots, x_n) \) subject to the \( k \) inequality constraints:

\[ g(x_1, x_2, \ldots, x_n) \leq b^*_1, \ldots, g(x_1, x_2, \ldots, x_n) \leq b^*_k \]

where \( b^* = (b^*_1, \ldots, b^*_k) \). Let \( x_1^*(b^*), \ldots, x_n^*(b^*) \) denote the optimal solution and let \( \lambda_1(b^*), \ldots, \lambda_k(b^*) \) be the corresponding Lagrange multipliers. Suppose that as \( b \) varies near \( b^* \), then \( x_1^*(b^*), \ldots, x_n^*(b^*) \) and \( \lambda_1(b^*), \ldots, \lambda_k(b^*) \) are differentiable functions and that \( x^*(b^*) \) satisfies the constraint qualification. Then for each \( j = 1, 2, \ldots, k \):

\[ \lambda_j(b^*) = \frac{\partial}{\partial b_j} f(x^*(b^*)). \]

**Proof.** (We consider the case of a binding constraint, and for simplicity, assume there is only one constraint, and that \( f \) and \( g \) are functions of two variables.) The Lagrangian is:

\[ L(x, y, \lambda; b) = f(x, y) - \lambda (g(x, y) - b). \]
The solution satisfies:

\[
0 = \frac{\partial L}{\partial x}(x^*(b), y^*(b), \lambda^*(b); b) \\
= \frac{\partial f}{\partial x}(x^*(b), y^*(b), \lambda^*(b)) \\
- \lambda^*(b) \frac{\partial h}{\partial x}(x^*(b), y^*(b), \lambda^*(b)) \\
\]

\[
0 = \frac{\partial L}{\partial y}(x^*(b), y^*(b), \lambda^*(b); b) \\
= \frac{\partial f}{\partial y}(x^*(b), y^*(b), \lambda^*(b)) \\
- \lambda^*(b) \frac{\partial h}{\partial y}(x^*(b), y^*(b), \lambda^*(b)) \\
\]

for all \( b \). Furthermore, since \( h(x^*(b), y^*(b)) = b \) for all \( b \):

\[
\frac{\partial h}{\partial x}(x^*, y^*) \frac{\partial x^*(b)}{\partial b} + \frac{\partial h}{\partial y}(x^*, y^*) \frac{\partial y^*(b)}{\partial b} = 1 \\
\]

for every \( b \). Therefore, using the chain rule, we have:

\[
\frac{df(x^*(b), y^*(b))}{db} = \frac{\partial f}{\partial x}(x^*, y^*) \frac{\partial x^*(b)}{\partial b} + \frac{\partial f}{\partial y}(x^*, y^*) \frac{\partial y^*(b)}{\partial b} \\
= \lambda^*(b) \left[ \frac{\partial h}{\partial x}(x^*, y^*) \frac{\partial x^*(b)}{\partial b} + \frac{\partial h}{\partial y}(x^*, y^*) \frac{\partial y^*(b)}{\partial b} \right] \\
= \lambda^*(b). \\
\]

The economic interpretation of the multiplier is as a ‘shadow’ price. For example, in the application for a firm maximising profits, it tells us how valuable another unit of input would be to the firm’s profits, or how much the maximum value changes for the firm when the constraint is relaxed. In other words, it is the maximum amount the firm would be willing to pay to acquire another unit of input.

### 2.14 The Envelope Theorem

Recall that:

\[
\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b) \\
\]

so that:

\[
\frac{d}{db} f(x(b), y(b); b) = \lambda(b) = \frac{\partial}{\partial b} L(x(b), y(b), \lambda(b); b). \\
\]

Hence, what we have found above is simply a particular case of the envelope theorem, which says that:

\[
\frac{d}{db} f(x(b), y(b); b) = \lambda(b) = \frac{\partial}{\partial b} \mathcal{L}(x(b), y(b), \lambda(b); b). \\
\]
Consider the problem of maximising \( f(x_1, x_2, \ldots, x_n) \) subject to the \( k \) inequality constraints:

\[
h_1(x_1, x_2, \ldots, x_n, c) = 0, \ldots, h_k(x_1, x_2, \ldots, x_n, c) = 0.
\]

Let \( x_1^*(c), \ldots, x_n^*(c) \) denote the optimal solution and let \( \mu_1(c), \ldots, \mu_k(c) \) be the corresponding Lagrange multipliers. Suppose that \( x_1^*(c), \ldots, x_n^*(c) \) and \( \mu_1(c), \ldots, \mu_k(c) \) are differentiable functions and that \( x^*(c) \) satisfies the constraint qualification. Then for each \( j = 1, 2, \ldots, k \):

\[
\frac{d}{dc} f(x^*(c); c) = \frac{\partial}{\partial c} \mathcal{L}(x^*(c), \mu(c); c).
\]

Note: if \( h_i(x_1, x_2, \ldots, x_n, c) = 0 \) can be expressed as some \( h'_i(x_1, x_2, \ldots, x_n) - c = 0 \), then we are back at the previous case, in which we have found that:

\[
\frac{d}{dc} f(x^*(c); c) = \frac{\partial}{\partial c} \mathcal{L}(x^*(c), \mu(c); c) = \lambda_j(c).
\]

But the statement is more general.

We will prove this for the simple case of an unconstrained problem. Let \( f(x; a) \) be a continuous function of \( x \in \mathbb{R}^n \) and the scalar \( a \). For any \( a \), consider the problem of finding \( \max f(x; a) \). Let \( x^*(a) \) be the maximiser which we assume is differentiable of \( a \). We will show that:

\[
\frac{d}{da} f(x^*(a); a) = \frac{\partial}{\partial a} f(x^*(a); a).
\]

Apply the chain rule:

\[
\frac{d}{da} f(x^*(a); a) = \sum_i \frac{\partial f}{\partial x_i}(x^*(a); a) \frac{\partial x_i^*}{\partial a}(a) + \frac{\partial f}{\partial a}(x^*(a); a)
\]

\[
= \frac{\partial f}{\partial a}(x^*(a); a)
\]

since \( \frac{\partial f}{\partial x_i}(x^*(a); a) = 0 \) for all \( i \) by the first order conditions. Intuitively, when we are already at a maximum, changing slightly the parameters of the problem or the constraints, does not affect the optimal solution (but it does affect the value at the optimal solution).

## 2.15 Solutions to activities

### Solution to Activity 2.1

(a) For any possible solution, \( x^* > 1 \), we have that \( x^* + 1 > 1 \) and \( \ln(x^* + 1) > \ln(x^*) \) and therefore \( x^* \) cannot be a solution. As \( \ln x \) is a strictly increasing function and the feasible set is unbounded there is no solution to this problem.

(b) For any possible solution, \( x^* < 1 \), we can choose an \( \epsilon \), where \( \epsilon < 1 - x^* \), such that we have that \( x^* + \epsilon < 1 \). But then we have \( \ln(x^* + \epsilon) > \ln(x^*) \) and therefore \( x^* \) cannot be a solution. As \( \ln x \) is a strictly increasing function and the feasible set is not closed there is not solution to this problem.

(c) As the feasible set is empty there is no solution to this problem.
2. Constrained optimisation: tools

Solution to Activity 2.2

(a) In this case there is no maximum as the feasible set is empty.

(b) In this case there is no maximum as the feasible set is unbounded.

(c) The maximum is achieved at $x = 1$.

(d) In this case there is no maximum as the feasible set is unbounded.

(e) The maximum is achieved at either $x = 1$ or $x = -1$.

(f) In this case there is no maximum as the feasible set is not closed around 2.

(g) The only feasible $x$ is $x = 1$ and therefore $x = 1$ yields the maximum.

(h) The maximum is achieved at $x = 2$. Note that this is true even though the feasible set is not closed.

Solution to Activity 2.3

In what follows refer to Figure 2.3. The profit function for this problem is $4y - c(y)$ and the problem calls for maximising this function with respect to $y$ subject to the constraint that $y \leq k$. To graph the set $B$ and to find the solution we look at the $(k,v)$ space. We first plot the function $v(k) = 4k - c(k)$. As can be seen in Figure 2.3 this function has an inverted $U$ shape which peaks at $k = 2$ with a maximum value of 4. Remember that the constraint is $y \leq k$ and therefore, whenever $k > 2$ we can always achieve the value 4 by choosing $y = 2 < k$. This is why the boundary of set $B$ does not follow the function $v(k) = 4k - c(k)$ as $k$ is larger than 2 but rather flattens out above $k = 2$. From the graph it is clear that for all $k \geq 2$ the solution is $y = 2$ and for all $0 \leq k < 2$ the solution is $y = k$. 

![Figure 2.3: Concave profit function in Activity 2.3.](image-url)
Solution to Activity 2.4

(a) $A \cup B$ is sometimes convex and sometimes not convex. For an example in which it is not convex consider the following sets in $\mathbb{R}$: $A = [0, 1]$ and $B = [2, 3]$. Consider $x = 1 \in A \subset A \cup B$ and $y = 2 \in B \subset A \cup B$ and $t = 0.5$. We now have $tx + (1 - t)y = 1.5 \notin [0, 1] \cup [2, 3]$.

(b) This set is always convex. To see this, take any two elements $x, y \in A + B$. By definition $x = x_A + x_B$ where $x_A \in A$ and $x_B \in B$. Similarly, $y = y_A + y_B$ where $y_A \in A$ and $y_B \in B$. Observe that for any $t \in [0, 1]$:

$$tx + (1 - t)y = tx_A + (1 - t)y_A + tx_B + (1 - t)y_B.$$

As $A$ and $B$ are convex, $tx_A + (1 - t)y_A \in A$ and $tx_B + (1 - t)y_B \in B$. But now we are done, as by the definition of $A + B$, $tx + (1 - t)y \in A + B$.

A reminder of your learning outcomes

By the end of this chapter, you should be able to:

- formulate a constrained optimisation problem
- discern whether you could use the Lagrange method to solve the problem
- use the Lagrange method to solve the problem, when this is possible
- discern whether you could use the Kuhn-Tucker Theorem to solve the problem
- use the Kuhn-Tucker Theorem to solve the problem, when this is possible
- discuss the economic interpretation of the Lagrange multipliers
- carry out simple comparative statics using the Envelope Theorem.

2.16 Sample examination questions

1. A household has a utility $u(x_1, x_2) = x_1^a x_2^b$ where $a, b > 0$ and $a + b = 1$. It cannot consume negative quantities of either good. It has strictly positive income $y$ and faces prices $p_1, p_2 > 0$. What is its optimal consumption bundle?

2. A household has a utility $u(x_1, x_2) = (x_1 + 1)^a (x_2 + 1)^b$ where $a, b > 0$ and $a + b = 1$. It cannot consume negative quantities of either good. It faces a budget with strictly positive income $y$ and prices $p_1, p_2 > 0$. What is its optimal consumption bundle?

2.17 Comments on the sample examination questions

1. The utility function $u(x_1, x_2)$ is real-valued if $x_1, x_2 \geq 0$. Therefore, we have:

$$Z = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}.$$
Our problem is to maximise $u(x_1, x_2)$ on $Z$ and subject to the budget constraint $p_1 x_1 + p_2 x_2 \leq y$.

The constraint function is differentiable and convex. The problem is that while the objective function, $u(x_1, x_2)$, is concave (see below), it is not differentiable at $(0, 0)$ as $Du(x_1, x_2) = (ax_1^{a-1} x_2^b, bx_1^{a-1} x_2^{b-1})$ is not well-defined at that point.

Note, however, that in a valid solution both $x_1$ or $x_2$ must be non-zero, for suppose that without loss of generality $x_1 = 0$. The marginal utility for good 1 is infinite, which means that trading a small amount of good 2 for good 1 would increase the utility. As a result the constraint $(x_1, x_2) \in Z$ should be slack at the maximum.

We next verify that the objective function is concave. For this we look at the Hessian:

$$D^2 u(x_1, x_2) = \begin{pmatrix} -abx_1^{a-2}x_2^b & abx_1^{a-1}x_2^{b-1} \\ abx_1^{a-1}x_2^{b-1} & -abx_1^{a-1}x_2^{b-2} \end{pmatrix}.$$ 

The principal minors of the first order are negative:

$$-abx_1^{a-2}x_2^b, -abx_1^{a-1}x_2^{b-2} < 0.$$ 

The second order minor is non-negative:

$$|A_2| = (a^2 b^2 - a^2 b^2) x_1^{2(a-1)} x_2^{2(b-1)} = 0.$$ 

Therefore, the Hessian is negative semi-definite and so $u(x_1, x_2)$ is concave. As for the constraint, it is a linear function and hence convex.

As prices and income are strictly positive, we can find strictly positive $x_1$ and $x_2$ that would satisfy the budget constraint with inequality. Therefore, we can use the Kuhn-Tucker Theorem to solve this problem. The Lagrangian is:

$$\mathcal{L} = x_1^a x_2^b + \lambda [y - p_1 x_1 - p_2 x_2].$$ 

The first order conditions are:

$$ax_1^{a-1} x_2^b = \lambda p_1,$$

$$bx_1^{a-1} x_2^{b-1} = \lambda p_2.$$ 

If $\lambda > 0$, dividing the two equations we have:

$$\frac{ax_2}{bx_1} = \frac{p_1}{p_2} \quad \text{(MRS = price ratio)}.$$ 

Plugging this into the constraint (that holds with equality when $\lambda > 0$),

$$x_1 = ay \frac{p_1}{p_1}, \quad x_2 = by \frac{p_2}{p_2}.$$ 

As all the conditions are satisfied, we are assured that this is a solution. To get that maximum value function, we plug the solution into the objective function to get:

$$v(p_1, p_2, y) = a^b b^a p_1^{-a} p_2^{-b}.$$ 

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2.17. Comments on the sample examination questions

2. The utility function \( u(x_1, x_2) \) is real-valued if \( x_1, x_2 \geq -1 \), but the consumer cannot consume negative amounts. Therefore, we have \( Z = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0\} \).

Our problem is to maximise \( u(x_1, x_2) \) on \( Z \) and subject to the budget constraint \( p_1 x_1 + p_2 x_2 \leq y \).

The constraint function is differentiable and convex. The objective function \( u(x_1, x_2) \) is concave (see below) and differentiable (see below).

We first make sure that the objective function is concave. For this we look at the Hessian:

\[
D^2u(x_1, x_2) = \begin{pmatrix}
-ab(x_1 + 1)^{a-2}(x_2 + 1)^b & ab(x_1 + 1)^{a-1}(x_2 + 1)^{b-1} \\
ab(x_1 + 1)^{a-1}(x_2 + 1)^{b-1} & -ab(x_1 + 1)^a(x_2 + 1)^{b-2}
\end{pmatrix}
\]

The principal minors of the first order are negative:

\[-ab(x_1 + 1)^{a-2}(x_2 + 1)^b, -ab(x_1 + 1)^a(x_2 + 1)^{b-2} < 0\]

and of the second order is non-negative:

\[|A_2| = (a^2b^2 - a^2b^2)(x_1 + 1)^{2(a-1)}(x_2 + 1)^{2(b-1)} = 0.\]

Therefore, the Hessian is negative semi-definite and so \( u(x_1, x_2) \) is concave. As for the constraint, it is a linear function and hence convex.

As prices and income are strictly positive, we can find strictly positive \( x_1 \) and \( x_2 \) that would satisfy the budget constraint with inequality. Therefore, we can use the Kuhn-Tucker Theorem to solve this problem. As in this problem \( x_i = 0 \) could be part of a solution, we can add the non-negativity constraints. The Lagrangian is:

\[ \mathcal{L} = x_1^a x_2^b + \lambda_0[y - p_1 x_1 - p_2 x_2] + \lambda_1 x_1 + \lambda_2 x_2. \]

The first order conditions are:

\[
a(x_1 + 1)^{a-1}(x_2 + 1)^b = \lambda p_1 + \lambda_1 \\
b(x_1 + 1)^a(x_2 + 1)^{b-1} = \lambda p_2 + \lambda_2.
\]

There may be three types of solutions (depending on the multipliers), one interior and two corner solutions. Assume first that \( \lambda_1 = \lambda_2 = 0 \) but that \( \lambda_0 > 0 \), dividing the two equations we have:

\[ \frac{a(x_2 + 1)}{b(x_1 + 1)} = \frac{p_1}{p_2} \quad \text{(MRS = price ratio)}. \]

Plugging this into the budget constraint (that holds with equality when \( \lambda_0 > 0 \)):

\[
x_1 = \frac{ay + ap_2 - bp_1}{p_1}, \quad x_2 = \frac{by - ap_2 + bp_1}{p_2} \\
\lambda_0 = a^a b^b p_1^{-1} p_2^{-b}.
\]

This is a solution only if both \( x_1 \) and \( x_2 \) are non-negative. This happens if and only if:

\[ y > \max \left( \frac{-ap_2 + bp_1}{a}, \frac{ap_2 - bp_1}{b} \right). \]
Consider now the case of $\lambda_1 > 0$ implying that $x_1 = 0$. In this case we must have that $x_2 = y/p_2$ implying that $\lambda_2 = 0$, as the consumer might as well spend all his income on good 2. From the first order conditions we have:

$$
\lambda_0 = \left( \frac{b}{p_2} \right) \left( \frac{y}{p_2} + 1 \right)^{-a}
$$

$$
\lambda_1 = \left( \frac{bp_1 - ap_2 - ay}{p_2} \right) \left( \frac{y}{p_2} + 1 \right)^{-a}.
$$

This is part of a solution only if $\lambda_1 \geq 0$, i.e. if and only if $y \leq (bp_1 - ap_2)/a$ and in that case the solution is $x_1 = 0$ and $x_2 = y/p_2$. Similarly, if $y \leq -(bp_1 + ap_2)/a$, the solution is $x_1 = y/p_1$ and $x_2 = 0$. 
Chapter 3
The consumer’s utility maximisation problem

3.1 Aim of the chapter

The aim of this chapter is to review the standard assumptions of consumer theory on preferences, budget sets and utility functions, treating them in a more sophisticated way than in intermediate microeconomics courses, and to understand the consequences of these assumptions through the application of constrained optimisation techniques.

3.2 Learning outcomes

By the end of this chapter, you should be able to:

- outline the assumptions made on consumer preferences
- describe the relationship between preferences and the utility function
- formulate the consumer’s problem in terms of preferences and in terms of utility maximisation
- define uncompensated demand, and to interpret it as the solution to a constrained optimisation problem
- define the indirect utility function, and to interpret it as a maximum value function.

3.3 Essential reading

This chapter is self-contained and therefore there is no essential reading assigned. But you may find further reading very helpful.

3.4 Further reading

For a reminder of the intermediate microeconomics treatment of consumer theory read:

- Varian, H.R. *Intermediate Microeconomics*. Chapters 2–6, or the relevant section of any intermediate microeconomics textbook.

For a more sophisticated treatment comparable to this chapter read:
3. The consumer's utility maximisation problem

- Varian, H.R. *Microeconomic Analysis*. Chapter 7 or the relevant section of any mathematical treatment of consumer theory.

3.5 Preferences

3.5.1 Preferences

Consumer theory starts with the idea of preferences. Consumers are modelled as having preferences over all points in the consumption set, which is a subset of $\mathbb{R}^n$. Points in $\mathbb{R}^n$ are vectors of the form $x = (x_1, x_2, \ldots, x_n)$ where $x_i$ is the consumption of good $i$. Most treatments of consumer theory, including this chapter, assume that the consumption set is $\mathbb{R}^{n+}$ that is the subset of $\mathbb{R}^n$ with $x_i \geq 0$ for $i = 1, \ldots, n$.

If $x$ and $y$ are points in $\mathbb{R}^{n+}$ that a consumer is choosing between:

- $x \succ y$ means that the consumer strictly prefers $x$ to $y$ so given a choice between $x$ and $y$ the consumer would definitely choose $y$.
- $x \sim y$ means that the consumer is indifferent between $x$ to $y$ so the consumer would be equally satisfied by either $x$ or $y$.
- $x \succeq y$ means that the consumer weakly prefers $x$ to $y$ that is either $x \succ y$ or $x \sim y$.

3.5.2 Assumptions on preferences

The most important assumption on preferences are:

- Preferences are **complete** if for any $x$ and $y$ in $\mathbb{R}^{n+}$ either $x \succeq y$ or $y \succeq x$.
- Preferences are **reflexive** that is for any $x$ in $\mathbb{R}^{n+}$ $x \succeq x$.
- Preferences are **transitive**, that is for $x$, $y$ and $z$ in $\mathbb{R}^{n+}$ $x \succeq y$ and $y \succ z$ implies that $x \succ z$.

A technical assumption on preferences is:

- Preferences are **continuous** if for all $y$ in $\mathbb{R}^{n+}$ the sets $\{x : x \in \mathbb{R}^{n+}, x \succeq y\}$ and $\{x : x \in \mathbb{R}^{n+}, y \succeq x\}$ are closed.

Closed sets can be defined in a number of different ways.

In addition we often assume:

- Preferences satisfy **nonsatiation** if for any $x$ and $y$ in $\mathbb{R}^{n+}$, $x \succ y$ that is $x_i > y_i$ for $i = 1, 2, \ldots, n$ implies that $x \succ y$.

Note that nonsatiation is not always plausible. Varian argues that if the two goods are chocolate cake and ice cream you could very plausibly be satiated. Some books refer to nonsatiation as ‘more is better’, others use the mathematical term ‘monotonicity’.

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Preferences satisfy **convexity**, that is for any \( y \) in \( \mathbb{R}^n^+ \) the set \( \{ x : x \in \mathbb{R}^n^+, x \succeq y \} \) of points preferred to \( y \) is **convex**.

Convex sets are defined in the previous chapter and on pages 70 and 71 of Chapter 6 of Dixit and Section 2.2 of Chapter 2 of Sydsæter et al. The convexity assumption is very important for the application of Lagrangian methods to consumer theory and we will come back to it. Some textbooks discuss convexity in terms of diminishing rates of marginal substitution, others describe indifference curves as being convex to the origin. Different textbooks work with slightly different forms of these assumptions; it does not matter, the resulting theory is the same.

**Activity 3.1** Assume there are two goods and that all the assumptions given above hold. Pick a point \( y \) in \( \mathbb{R}^2^+ \).

(a) Show the set \( \{ x : x \in \mathbb{R}^n^+, x \sim y \} \). What is this set called?

(b) What is the marginal rate of substitution?

(c) Show in a diagram the set of points weakly preferred to \( y \), that is \( \{ x : x \in \mathbb{R}^n^+, x \succeq y \} \). Make sure that it is convex. What does this imply for the marginal rate of substitution?

(d) Show the set of points \( \{ x : x \in \mathbb{R}^n^+, x \geq y \} \), where \( x \geq y \) means that \( x_i \geq y_i \) for \( i = 1, 2, \ldots, n \). Explain why nonsatiation implies that this set is a subset of the set of points weakly preferred to \( y \), that is \( \{ x : x \in \mathbb{R}^n^+, x \succeq y \} \).

(e) Explain why the nonsatiation assumption implies that the indifference curve cannot slope upwards.

### 3.6 The consumer’s budget

#### 3.6.1 Definitions

**Assumption.** The consumer’s choices are constrained by the conditions that:

\[
p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq m
\]

\[
x \in \mathbb{R}^n^+
\]

where \( p_i \) is the price of good \( i \) and \( m \) is income. We assume that prices and income are strictly positive, that is \( p_i > 0 \) for \( i = 1, \ldots, n \) and \( m > 0 \).

**Definition 13** The **budget constraint** is the inequality:

\[
p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq m.
\]

The budget constraint is sometimes written as \( p x \leq m \) where \( p \) is the vector prices \((p_1 \cdots p_n)\) and \( px \) is notation for the sum \( p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \).
3. The consumer’s utility maximisation problem

**Definition 14**

The **budget line** is the set of points with \( p_1x_1 + p_2x_2 + \cdots + p_nx_n = m \) and \( x_i \geq 0 \) for \( i = 1, \ldots, n \).

**Definition 15**

The **budget set** is the subset of points in \( \mathbb{R}^n^+ \) satisfying the budget constraint. In formal notation it is:

\[
\{ x : x \in \mathbb{R}^n^+, \ p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq m \}.
\]

**Activity 3.2**

Assume that \( n = 2 \) and draw a diagram showing:

(a) the budget line and its gradient

(b) the points where the budget line meets the axes

(c) the budget set.

**Activity 3.3**

Draw diagrams showing the effects on the budget line and budget set of the following changes:

(a) an increase in \( p_1 \)

(b) a decrease in \( p_2 \)

(c) an increase in \( m \).

3.7 Preferences and the utility function

3.7.1 The consumer’s problem in terms of preferences

**Definition 16**

The consumer’s problem stated in terms of preferences is to find a point \( x^* \) in \( \mathbb{R}^n^+ \) with the property that \( x^* \) satisfies the budget constraint, so:

\[
p_1x_1^* + p_2x_2^* + \cdots + p_nx_n^* \leq m
\]

and:

\[
x^* \succeq x
\]

for all \( x \) in \( \mathbb{R}^n^+ \) that satisfy the budget constraint.
3.7. Preferences and utility

The difficulty with expressing the consumer’s problem in terms of preferences is that we have no techniques for solving the problem. However preferences can be represented by utility functions, which makes it possible to define the consumer’s problem in terms of constrained optimisation and solve it using the tools developed in the previous chapter.

**Definition 17**

A utility function \( u(x) \) which takes \( \mathbb{R}^{n^+} \) into \( \mathbb{R} \) represents preferences \( \succeq \) if:

- \( u(x) > u(y) \) implies that \( x \succ y \)
- \( u(x) = u(y) \) implies that \( x \sim y \)
- \( u(x) \geq u(y) \) implies that \( x \succeq y \).

The following result and its proof are beyond the scope of this course, but the result is worth knowing.

**Theorem 6**

If preferences satisfy the completeness, transitivity and continuity assumptions they can be represented by a continuous utility function.

Varian’s *Microeconomic Analysis* proves a weaker result on the existence of a utility function in Chapter 7.1. Again this is beyond the scope of this course.

3.7.3 Cardinal and ordinal utility

One of the standard points made about consumer theory is that the same set of consumer preferences can be represented by different utility functions so long as the order of the numbers on indifference curves is not changed. For example if three indifference curves have utility levels 1, 2 and 3, replacing 1, 2 and 3 by 2, 62 and 600 does not change the preferences being represented because \( 1 < 2 < 3 \) and \( 2 < 62 < 600 \). However replacing 1, 2 and 3 by 62, 2 and 600 does change the preferences being represented because it is not true that \( 62 < 2 < 600 \). The language used to describe this is that the utility function is ordinal rather than cardinal, so saying that one bundle of goods gives higher utility than another means something, but saying that one bundle gives twice as much utility as the other is meaningless.

This can be stated more formally.

**Theorem 7**

If the utility function \( u(x) \) and the function \( b(x) \) are related by \( u(x) = a(b(x)) \) where \( a \) is a strictly increasing function, then \( b(x) \) is also a utility function representing the same preferences as \( u(x) \).
As there are many different strictly increasing functions this result implies that any set of preferences can be represented by many different utility functions.

Proof. As \( a \) is strictly increasing and \( u \) represents preferences:

- \( b(x) > b(y) \) implies that \( u(x) > u(y) \) which implies \( x > y \)
- \( b(x) = b(y) \) implies that \( u(x) = u(y) \) which implies \( x \sim y \)
- \( b(x) \geq b(y) \) implies that \( u(x) \geq u(y) \) which implies \( x \succeq y \), so \( b(x) \) is a utility function representing the same preferences as \( u(x) \).

Activity 3.4  Suppose that \( f \) is a function taking \( \mathbb{R}^{n+} \) into \( \mathbb{R} \), and the term \( \geq_f \) is defined by \( f(x) \geq f(y) \) implies that \( x \geq_f y \). Explain why the relationship given by \( \geq_f \) is complete, reflexive and transitive.

Activity 3.5  Suppose that the utility function is differentiable, and that:

\[
\frac{\partial u(x)}{\partial x_i} > 0 \quad \text{for} \ i = 1, \ldots, n.
\]

Explain why this implies that the preferences satisfy nonsatiation.

3.8  The consumer’s problem in terms of utility

3.8.1  Uncompensated demand and the indirect utility function

Definition 18

The consumer’s utility maximisation problem is:

\[
\text{max } u(x) \quad \text{s.t. } p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq m \quad x \in \mathbb{R}^{n+}.
\]

Note that this has the same form as the constrained optimisation problem in the previous chapter. The previous chapter showed you how to solve the consumer’s problem with the Cobb-Douglas utility function \( u(x_1, x_2) = x_1^a x_2^b \) and the utility function \( u(x_1, x_2) = (x_1 + 1)^a(x_2 + 1)^b \).

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3.8. The consumer’s problem in terms of utility

The solution to the consumer’s utility maximisation problem is **uncompensated demand**. It depends upon prices \( p_1, p_2, \ldots, p_n \) and income \( m \). The uncompensated demand for good \( i \) is written as:

\[
x_i(p_1, p_2, \ldots, p_n, m)
\]

or using vector notation as \( x_i(p, m) \) where \( p \) is the vector \( (p_1, p_2, \ldots, p_n) \).

Notation \( x(p, m) \) is used for the vector of uncompensated demand \( (x_1(p, m), x_2(p, m), \ldots, x_n(p, m)) \).

**Definition 19**

The notation \( x(p, m) \) suggests that uncompensated demand is a function of \( (p, m) \), which requires that for each value of \( (p, m) \) there is only one solution to the utility maximising problem. This is usually so in the examples economists work with, but it does not have to be, and we will look at the case with a linear utility function where the consumer’s problem has multiple solutions for some values of \( p \).

Recall from Chapter 1 the definition of the maximum value function as the value of the function being maximised at the solution to the problem.

The **indirect utility function** is the maximum value function for the consumer’s utility maximisation problem. It is written as:

\[
v(p_1, p_2, \ldots, p_n, m)
\]

or in vector notation \( v(p, m) \) where:

\[
v(p, m) = u(x(p, m)).
\]

There are two important results linking preferences and utility functions.

**Theorem 8**

If preferences are presented by a utility function \( u(x) \) then the solutions to the consumer’s utility maximising function in terms of preferences are the same as the solutions to the consumer’s problem in terms of utility.

**Theorem 9**

If two utility functions represent the same preferences the solutions to the consumer’s utility maximisation problem are the same with the two utility functions but the indirect utility functions are different.

These results are not difficult to prove, the next learning activity asks you to do this.
3. The consumer’s utility maximisation problem

3.8.2 Nonsatiation and uncompensated demand

The nonsatiation assumption on preferences is that for any \( x \) and \( y \) in \( \mathbb{R}^n + \), if \( x > y \), that is \( x_i \geq y_i \) for all \( i \) and \( x_i > y_i \) for some \( i \), then \( x > y \). If the preferences are represented by a utility function \( u \), this requires that if \( x > y \) then \( u(x) > u(y) \), so the utility function is strictly increasing in the consumption of at least one good, and non-decreasing in consumption of any good. In intuitive terms, the assumption says that more is better, so a consumer will spend all available income.

**Proposition 6**

If the nonsatiation condition is satisfied any solution to the consumer’s utility maximising problem satisfies the budget constraint as an equality.

**Proof.** To see this, suppose that the budget constraint is not satisfied as an equality so:

\[
p_1 x_1 + p_2 x_2 + \cdots + p_n x_n < m
\]

and increasing consumption of good \( i \) increases utility so if \( \epsilon > 0 \):

\[
u(x_1, x_2, \ldots, x_i + \epsilon, \ldots, x_n) > u(x_1, x_2, \ldots, x_i, \ldots, x_n).
\]

As \( p_1 x_1 + p_2 x_2 + \cdots + p_n x_n < m \) if \( \epsilon \) is small enough the point \((x_1, x_2, \ldots, x_i + \epsilon, \ldots, x_n)\) satisfies the budget constraint and gives higher utility that \((x_1, x_2, \ldots, x_i, \ldots, x_n)\) so \((x_1, x_2, \ldots, x_i, \ldots, x_n)\) cannot solve the consumer’s problem. 

| Activity 3.6 | Explain why Theorem 8 holds. |
| Activity 3.7 | Explain why Theorem 9 holds. |
| Activity 3.8 | Find the indirect utility function for the Cobb-Douglas utility function \( u(x_1, x_2) = x_1^a x_2^b \). |
| Activity 3.9 | The objective of this activity is to solve the consumer’s utility maximising function with a linear utility function \( 2x_1 + x_2 \) subject to the constraints \( p_1 x_1 + p_2 x_2 \leq m \), \( x_1 \geq 0 \) and \( x_2 \geq 0 \). Assume that \( p_1 > 0 \), \( p_2 > 0 \) and \( m > 0 \). |
| (a) | Draw indifference curves for the utility function \( u(x_1, x_2) = 2x_1 + x_2 \). What is the marginal rate of substitution? |
| (b) | Assume that \( p_1/p_2 < 2 \). Use your graph to guess the values of \( x_1 \) and \( x_2 \) that solve the maximising problem. Which constraints bind, that is have \( h_i(x_1, x_2) = k_i \) where \( h_i \) is constraint function \( i \). Which constraints do not bind, that is have \( h_i(x_1, x_2) < k_i \)? Does the problem have more than one solution? |
| (c) | Assume that \( p_1/p_2 = 2 \). Use your graph to guess the values of \( x_1 \) and \( x_2 \) that solve the maximising problem. Which constraints bind? Which constraints do not bind? Is there more than one solution? |
3.9 Solution to activities

Solution to Activity 3.1

Assume that the consumer’s preferences satisfy all the assumptions of Section 3.1. Refer to Figure 3.1.

(a) The set \( \{x : x \in \mathbb{R}^n, x \sim y \} \) is the indifference curve.

(b) The marginal rate of substitution is the gradient of the indifference curve.

(c) The set \( \{x : x \in \mathbb{R}^n, x \succeq y \} \) is the entire shaded area in Figure 3.1. The marginal rate of substitution decreases as \( x_1 \) increases because this set is convex.

(d) The set \( \{x : x \in \mathbb{R}^n, x \succeq y \} \), where \( x \succeq y \) means that \( x_i \geq y_i \) for \( i = 1, 2, \ldots, n \) is the lightly shaded area in Figure 3.1. The nonsatiation assumption states that all points in this set are weakly preferred to \( y \), that is \( x \succeq y \).

(e) If the indifference curve slopes upwards there are points on the indifference curve \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) with \( y_1 > x_1 \) and \( y_2 > x_2 \) so \( x \gg y \). Nonsatiation then implies that \( x \succ y \) which is impossible if \( x \) and \( y \) are on the same indifference curve.

Solution to Activity 3.2

The budget line in Figure 3.2 has gradient \(-p_1/p_2\) and meets the axes at \( m/p_1 \) and \( m/p_2 \).

Solution to Activity 3.3

Refer to Figure 3.3:
3. The consumer’s utility maximisation problem

Figure 3.1: Indifference curves and preference relationships.

Figure 3.2: The budget set and budget line.
Figure 3.3: The effect of changes in prices and income on the budget line.

- An increase in $p_1$ to $p'_1$ shifts the budget line from $CE$ to $CD$. The budget line becomes steeper.

- A decrease in $p_2$ to $p'_2$ shifts the budget line from $CE$ to $BE$. The budget line becomes steeper.

- An increase in $m$ to $m'$ shifts the budget line out from $CE$ to $AF$. The gradient of the budget line does not change.

Solution to Activity 3.4

This activity tells you conditions on the utility function that ensure that preferences have various properties, and asks you to explain why.

Suppose that $f$ is a function taking $\mathbb{R}^{n+}$ into $\mathbb{R}$, and the term $\geq_f$ is defined by $f(x) \geq f(y)$ implies that $x \geq_f y$.

The values of the functions $f(x)$ are real numbers. The relationship $\geq$ for real numbers is complete, so for any $x$ and $y$ either $f(x) \geq f(y)$ or $f(y) \geq f(x)$.

The relationship $\geq$ for real numbers is reflexive so for any $x$, $f(x) \geq f(x)$.

The relationship $\geq$ is transitive, so for any $x$, $y$ and $z$, $f(x) \geq f(y)$ and $f(y) \geq f(z)$ implies that $f(x) \geq f(z)$.

Thus, as $f(x) \geq f(y)$ implies that $x \geq_f y$, the relationship $\geq_f$ is complete, reflexive and transitive.
3. The consumer’s utility maximisation problem

Solution to Activity 3.5

As:

\[
\frac{\partial u(x)}{\partial x_i} > 0 \quad \text{for} \quad i = 1, \ldots, n
\]

if \( y_i > x_i \) for \( i = 1, \ldots, n \) then \( u(y) > u(x) \), so \( y \succ x \), and nonsatiation is satisfied.

Solution to Activity 3.6

Theorem 8 states that if preferences are represented by a utility function \( u(x) \) then the solutions to the consumer’s utility maximising function in terms of preferences are the same as the solutions to the consumer’s problem in terms of utility.

To see why suppose that a point \( x^* \) solves the consumer’s problem in terms of preferences so \( x^* \succeq x \) for all points in \( \mathbb{R}^{n+} \) that satisfy the budget constraint. A point \( x^* \) solves the consumer’s utility maximisation problem if \( u(x^*) \geq u(x) \) for all points in \( \mathbb{R}^{n+} \) that satisfy the budget constraint. If the utility function represents the preferences then the set of points for which \( x^* \succeq x \) is the same as the set of points for which \( u(x^*) \geq u(x) \) is the same as the set of points for which \( u(x^*) \geq u(x) \), so the two problems have the same solutions.

Solution to Activity 3.7

Theorem 9 states that if two utility functions represent the same preferences the solutions to the consumer’s utility maximisation problem are the same with the two utility functions but the indirect utility functions are different.

This follows directly from Theorem 8, because if \( u(x) \) and \( u(x^*) \) represent the same preferences the solutions to the consumer’s utility maximisation with the two utility functions are the same as the solution to the consumer’s problem in terms of the preferences and so are the same as each other.

Solution to Activity 3.8

The Sample examination question for Chapter 2 gave the solution to the consumer’s Cobb-Douglas utility function \( u(x_1, x_2) = x_1^a x_2^b \) where \( a > 0 \), \( b > 0 \) and \( a + b = 1 \). The solutions are \( x_1 = a(m/p_1) \) and \( x_2 = b(m/p_2) \) and so the uncompensated demands are:

\[
\begin{align*}
x_1(p_1, p_2, m) &= \frac{m}{p_1} \\
x_2(p_1, p_2, m) &= \frac{m}{p_2}
\end{align*}
\]
3.9. Solution to activities

The objective of this activity is to solve the consumer’s utility maximising function with a linear utility function $2x_1 + x_2$ subject to the constraints $p_1x_1 + p_2x_2 \leq m$, $x_1 \geq 0$ and $x_2 \geq 0$. Assume that $p_1 > 0$, $p_2 > 0$ and $m > 0$.

(a) The indifference curves for the utility function $u(x_1, x_2) = 2x_1 + x_2$ are parallel straight lines with gradient $-2$ as shown in Figure 3.4. The marginal rate of substitution is 2.

(b) From Figure 3.4 if $p_1/p_2 < 2$ the budget line is less steep than the indifference curves. The solution is at $x_1 = m/p_1$, $x_2 = 0$. The constraints $p_1x_1 + p_2x_2 \leq m$ and $x_2 \geq 0$ bind. The constraint $x_1 \geq 0$ does not bind. There is only one solution.

(c) From Figure 3.5 if $p_1/p_2 = 2$ the budget line is parallel to all the indifference curves, and one indifference curve coincides with the budget line. Any $(x_1, x_2)$ with $x_1 \geq 0$, $x_2 \geq 0$ and $p_1x_1 + p_2x_2 = m$ solves the problem. If $x_1 = 0$ the constraints $x_1 \geq 0$ and $p_1x_1 + p_2x_2 \leq m$ bind and the constraint $x_2 \geq 0$ does not bind. If $x_2 = 0$ the constraints $x_2 \geq 0$ and $p_1x_1 + p_2x_2 = m$ bind and the constraint $x_1 \geq 0$ does not bind. If $x_1 > 0$ and $x_2 > 0$ the constraint $p_1x_1 + p_2x_2 \leq m$ binds and the constraints $x_1 \geq 0$ and $x_2 \geq 0$ do not bind. There are many solutions.
From Figure 3.6 if \( p_1/p_2 > 2 \) the budget line is steeper than the indifference curves. The solution is at \( x_1 = 0, x_2 = m/p_2 \). The constraints \( p_1 x_1 + p_2 x_2 \leq m \) and \( x_1 \geq 0 \) bind. The constraint \( x_2 \geq 0 \) does not bind. There is only one solution.

The objective \( 2x_1 + x_2 \) is linear so concave, the constraint functions \( p_1 x_2 + p_2 x_2, x_1 \) and \( x_2 \) are linear so convex, the constraint qualification is satisfied because there are points that satisfy the constraints as strict inequalities, for example \( x_1 = m/2p_1, x_2 = m/2p_2 \). Thus the Kuhn-Tucker conditions are necessary and sufficient for a solution to this problem.

The Lagrangian is:

\[
L = 2x_1 + x_2 + \lambda_0(m - p_1 x_1 - p_2 x_2) + \lambda_1 x_1 + \lambda_2 x_2 \\
= x_1(2 - \lambda_0 p_1 + \lambda_1) + x_2(1 - \lambda_0 p_2 + \lambda_2).
\]

The first order conditions require that:

\[
2 = \lambda_0 p_1 - \lambda_1 \\
1 = \lambda_0 p_2 - \lambda_2.
\]

Checking for solutions with \( x_1 > 0 \) and \( x_2 = 0 \), complementary slackness forces \( \lambda_1 = 0 \), so \( \lambda_0 = 2/p_1 > 0 \) and \( \lambda_2 = \lambda_0 p_2 - 1 = 2p_2/p_1 - 1 \). Thus non-negativity of multipliers forces \( p_1/p_2 \leq 2 \). Complementary slackness forces \( x_1 = m/p_1 \). The point \( x_1 = m/p_1, x_2 = 0 \) is feasible. Thus if \( p_1/p_2 \leq 2 \) the point \( x_1 = m/p_1, x_2 = 0 \) solves the problem. If \( p_1/p_2 < 0 \) this is the unique solution.

Checking for solutions with \( x_1 = 0 \) and \( x_2 > 0 \), complementary slackness forces \( \lambda_2 = 0 \), so \( \lambda_0 = 1/p_2 \) and \( \lambda_1 = \lambda_0 p_1 - 2 = p_1/p_2 - 2 \). Thus non-negativity of multipliers forces \( p_1/p_2 \geq 2 \). Complementary slackness forces \( x_2 = m/p_2 \). The point \( x_1 = 0, x_2 = m/p_2 \) is feasible. Thus if \( p_1/p_2 \geq 2 \) the point \( x_1 = 0, x_2 = m/p_2 \) solves the problem. If \( p_2 \leq p_1/2 \) this is the only solution.

If \( p_1/p_2 = 2 \), and \( \lambda_0 = 2/p_2 = 1/p_2 \), then \( \lambda_1 = \lambda_2 = 0 \), so solutions with \( x_1 > 0 \) and \( x_2 > 0 \) are possible. As \( \lambda_0 > 0 \) complementary slackness forces \( p_1 x_1 + p_2 x_2 = m \). In this case there are multiple solutions.

**Solution to Activity 3.10**

Finding the indirect utility function for the utility function \( u(x_1, x_2) = 2x_1 + x_2 \), if \( p_1/p_2 < 2 \) then \( x_1(p_1, p_2, m) = m/p_1 \) and \( x_2(p_1, p_2, m) = 0 \) so

\[
v(p_1, p_2, m) = 2x_1 + x_2 = 2m/p_1.
\]

If \( p_1/p_2 = 2 \), any \( x_1, x_2 \) with \( x_1 \geq 0, x_2 \geq 0 \) and \( x_2 = (m - p_1 x_1)/p_2 \) solves the problem, so:

\[
v(p_1, p_2, m) = 2x_1 + \frac{(m - p_1 x_1)}{p_2} = \frac{m}{p_1} = \frac{2m}{p_1}.
\]

If \( p_1/p_2 > 2 \), then \( x_1(p_1, p_2, m) = 0 \) and \( x_2(p_1, p_2, m) = m/p_2 \) so

\[
v(p_1, p_2, m) = 2x_1 + x_2 = m/p_2.
\]

This is a perfectly acceptable answer to the question, but the indirect utility function can also be written more concisely as:

\[
v(p_1, p_2, m) = \max \left( \frac{2m}{p_1}, \frac{m}{p_2} \right).
\]
3.9. Solution to activities

Figure 3.5: Utility maximisation with \( u(x_1, x_2) = 2x_1 + x_2 \) and \( \frac{p_1}{p_2} = 2 \).

Figure 3.6: Utility maximisation with \( u(x_1, x_2) = 2x_1 + x_2 \) and \( \frac{p_1}{p_2} > 2 \).
3. The consumer’s utility maximisation problem

**Learning outcomes**

By the end of this chapter, you should be able to:

- outline the assumptions made on consumer preferences
- describe the relationship between preferences and the utility function
- formulate the consumer’s problem in terms of preferences and in terms of utility maximisation
- define uncompensated demand, and to interpret it as the solution to a constrained optimisation problem
- define the indirect utility function, and to interpret it as a maximum value function.

### 3.10 Sample examination questions

When answering these and any other examination questions be sure to explain why your answers are true, as well as giving the answer. For example here when you use Lagrangian techniques in Question 2 check that the problem you are solving satisfies the conditions of the theorem you are using, and explain your reasoning.

1. (a) Explain the meaning of the nonsatiation and transtivity assumptions in consumer theory.

   (b) Suppose that preferences can be represented by a utility function \( u(x_1, x_2) \). Do these preferences satisfy the transitivity assumption?

   (c) Assume that the derivatives of the utility function \( u(x_1, x_2) \) are strictly positive. Do the preferences represented by this utility function satisfy the nonsatiation assumption?

   (d) Continue to assume that the derivatives of the utility function \( u(x_1, x_2) \) are strictly positive. Is it possible that the solution to the consumer’s utility maximisation problem does not satisfy the budget constraint as an equality?

2. (a) Define uncompensated demand.

   (b) Define the indirect utility function.

   (c) A consumer has a utility function \( u(x_1, x_2) = x_1^\rho + x_2^\rho \) where \( 0 < \rho < 1 \). Solve the consumer’s utility maximisation problem.

   (d) What is the consumer’s uncompensated demand function for this utility function?

   (e) What is the consumer’s indirect utility function for this utility function?
3.11 Comments on sample examination questions

As with any examination question you need to explain what you are doing and why.

2. (d) The uncompensated demand functions for the consumer’s utility maximisation problem with \( u(x_1, x_2) = x_1^\rho + x_2^\rho \) are:

\[
x_1(p_1, p_2, m) = \left( \frac{-\frac{1}{\rho} - p_1}{p_1^{-\frac{1}{\rho}} + p_2^{-\frac{1}{\rho}}} \right) m
\]

\[
x_2(p_1, p_2, m) = \left( \frac{-\frac{1}{\rho} - p_2}{p_1^{-\frac{1}{\rho}} + p_2^{-\frac{1}{\rho}}} \right) m.
\]

(e) The indirect utility function is:

\[
v(p_1, p_2, m) = \left( p_1^{-\frac{\rho}{1-\rho}} + p_2^{-\frac{\rho}{1-\rho}} \right)^{1-\rho} m^\rho.
\]
3. The consumer’s utility maximisation problem
Chapter 4
Homogeneous and homothetic functions in consumer choice theory

4.1 Aim of the chapter

The aim of this chapter is to review the important concepts of a homogeneous function and introduce the idea of a homothetic function. Homogeneity is important for consumer theory because for any utility function the uncompensated demand and indirect utility functions are homogeneous of degree zero in prices and income. Homotheticity is a weaker concept than homogeneity. Assuming that the utility function is either homogeneous or homothetic has very strong implications for indifference curves, marginal rates of substitution and uncompensated demand which are explored in this chapter.

4.2 Learning outcomes

By the end of this chapter, you should be able to:

- state the definitions of homogeneous and homothetic functions
- explain why the uncompensated demand and indirect utility functions are homogeneous of degree zero in prices and income
- determine when the indirect utility function is increasing or non-decreasing in income and decreasing or non-increasing in prices
- explain the relationship between the homogeneity of a function and its derivative
- describe the implications of homogeneous and homothetic utility functions for the shape of indifference curves, the marginal rate of substitution and uncompensated demand.

4.3 Essential reading

This chapter is self-contained and therefore there is no essential reading assigned.
4. Homogeneous and homothetic functions in consumer choice theory

4.4 Further reading

- Varian, H.R. *Microeconomic Analysis*. Chapter 7 covers the indirect utility function. There is a brief discussion of homogeneous and homothetic production functions in Chapter 1.

Any textbook covering multivariate calculus will have some discussion of homogeneous functions.

4.5 Homogeneous functions

4.5.1 Definition

A function \( f(x_1, x_2, \ldots, x_n) \) taking \( \mathbb{R}^n_+ \) into \( \mathbb{R} \) is homogeneous of degree \( d \) in \((x_1, x_2, \ldots, x_n)\) if for all \((x_1, x_2, \ldots, x_n)\) in \( \mathbb{R}^n_+ \) and all numbers \( s > 0 \):

\[
f(sx_1, sx_2, \ldots, sx_n) = s^d f(x_1, x_2, \ldots, x_n).
\]

Using vector notation \( \mathbf{x} \) for \((x_1, x_2, \ldots, x_n)\), so \((sx_1, sx_2, \ldots, sx_n) = s\mathbf{x}\), this equation can be written as:

\[
f(s\mathbf{x}) = s^d f(\mathbf{x}).
\]

4.5.2 Homogeneity of degrees zero and one

The most important cases are homogeneity of degree zero and homogeneity of degree one. As \( s^0 = 1 \), \( f(x_1, x_2, \ldots, x_n) \) is homogeneous of degree zero if for all \( s > 0 \):

\[
f(sx_1, sx_2, \ldots, sx_n) = f(x_1, x_2, \ldots, x_n)
\]

so multiplying all components of \((x_1, x_2, \ldots, x_n)\) by the same number \( s > 0 \) has no effect on the value of the function. Using vector notation this says that:

\[
f(s\mathbf{x}) = f(\mathbf{x}).
\]

As \( s^1 = s \) a function is homogeneous of degree one if for all \( s > 0 \):

\[
f(sx_1, sx_2, \ldots, sx_n) = sf(x_1, x_2, \ldots, x_n)
\]

so multiplying all components of \((x_1, x_2, \ldots, x_n)\) by \( s \) has the effect of multiplying the function by \( s \). Using vector notation this equation says that:

\[
f(s\mathbf{x}) = sf(\mathbf{x}).
\]
Activity 4.1 For each of the following functions state whether or not it is homogeneous in \((x_1, x_2)\). If the function is homogeneous say of what degree.

(a) \(ax_1 + bx_2\)

(b) \((ax_1 + bx_2)^2\)

(c) \(ax_1^2 + bx_2\)

(d) \(cx_1^3 + dx_1^2x_2 + ex_1x_2^2 + fx_1^3\)

(e) \((cx_1^3 + dx_1^2x_2 + ex_1x_2^2 + fx_1^3)/(ax_1 + bx_2)^3\)

(f) \(x_1^αx_2 - 2β\). Assume that \(α > 0\) and \(β > 0\), do not assume that \(α + β = 1\).

(g) \(c\ln x_1 + d\ln x_2\).

4.6 Homogeneity, uncompensated demand and the indirect utility function

4.6.1 Homogeneity

Uncompensated demand and the indirect utility function are both homogeneous of degree zero in prices and income, that is for uncompensated demand for any number \(s > 0\):

\[x_i(sp_1, sp_2, \ldots, sp_n, sm) = x_i(p_1, p_2, \ldots, p_n, m) \quad \text{for} \quad i = 1, \ldots, n\]

or in vector notation:

\[\mathbf{x}(sp, m) = \mathbf{x}(sp, m)\]

and for the indirect utility function:

\[v(sp_1, sp_2, \ldots, sp_n, sm) = v(p_1, p_2, \ldots, p_n, m)\]

or in vector notation:

\[v(sp, sm) = v(p, m)\]

In words this says that when all prices and income are multiplied by the same positive number \(s\) the solution to the consumer’s utility problem does not change and neither does the level of utility. The next learning activity asks you to explain why this is so.
4. Homogeneous and homothetic functions in consumer choice theory

4.6.2 Other properties of indirect utility functions

Monotonicity in income

**Proposition 7**

For any utility function the indirect utility function is non-decreasing in income \( m \). With nonsatiation the indirect utility function is strictly increasing in \( m \).

**Proof.** From Chapter 1 a maximum value function is non-decreasing in the level of the constraint \( k \). For the consumer’s utility maximisation problem the maximum value function is the indirect utility function \( v(p_1, p_2, m) \), and the level of the constraint is \( m \), so the general result implies at once that indirect utility is non-decreasing in \( m \). Now assume that the consumer is not satiated and income rises from \( m \) to \( m' \). Nonsatiation implies that:

\[
p_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m) = m
\]

so:

\[
p_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m) < m'
\]

making it possible to increase both \( x_1 \) and \( x_2 \) by a small amount to \( x'_1 \) and \( x'_2 \) without violating the budget constraint so:

\[
p_1 x'_1 + p_2 x'_2 \leq m'
\]

and because of nonsatiation:

\[
u(x'_1, x'_2) > u(x_1(p_1, p_2, m), x_2(p_1, p_2, m)) = v(p_1, p_2, m).
\]

As \((x'_1, x'_2)\) is feasible with income \( m' \), the solution to the consumer’s problem with income \( m \) must give utility \( v(p_1, p_2, m) \) which cannot be smaller than \( u(x'_1, x'_2) \), so:

\[
v(p_1, p_2, m') \geq u(x'_1, x'_2).
\]

Taken together the last two inequalities imply that:

\[
v(p_1, p_2, m') > v(p_1, p_2, m).
\]

■

Monotonicity in prices

**Proposition 8**

With any utility function the indirect utility function is non-increasing in prices. With nonsatiation the indirect utility function is strictly decreasing in the prices of good \( i \) if the consumer purchases a strictly positive amount of the good, that is \( x_i(p_1, p_2, m) > 0 \).
Proof. Suppose that the price \( p_1 \) of good 1 falls from \( p_1 \) to \( p'_1 \). As \( x_1(p_1,p_2,m) \geq 0 \) and \( p'_1 < p_1 \) this implies that:

\[
p'_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m) \\
\leq p_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m).
\]

Thus \((x_1(p_1, p_2, m), x_2(p_1, p_2, m))\) is feasible with prices \( p'_1 \) and \( p_2 \) and income \( m \) so gives utility no greater than the utility from the solution to the consumer’s problem with prices \( p'_1 \) and \( p_2 \) and income \( m \) which is \( v(p'_1, p_2, m) \) so:

\[
v(p'_1, p_2, m) \geq u(x_1(p_1, p_2, m), x_2(p_1, p_2, m)) = v(p_1, p_2, m).
\]

Thus if \( p'_1 < p_1 \) then \( v(p'_1, p_2, m) \geq v(p_1, p_2, m) \). Now suppose that \( x_1(p_1, p_2, m) > 0 \) and the consumer is not satiated which implies that:

\[
p'_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m) \\
< p_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m) = m.
\]

Then at prices \( p'_1 \) and \( p_2 \) and income \( m \) there is a feasible point \((x'_1, x'_2)\) with \( x'_1 > x_1(p_1, p_2, m) \) and \( x'_2 > x_2(p_1, p_2, m) \) so:

\[
v(p'_1, p_2, m) \geq u(x'_1, x'_2).
\]

Nonsatiation implies that:

\[
u(x'_1, x'_2) > u(x_1(p_1, p_2, m), x_2(p_1, p_2, m)) = v(p_1, p_2, m).
\]

From the last two inequalities \( v(p'_1, p_2, m) > v(p_1, p_2, m) \). A similar argument applies to good 2. ■

**Activity 4.2** Consider the two goods case.

(a) What happens to the budget set and the budget line when \( p_1, p_2 \) and \( m \) are all multiplied by 2?

(b) What happens to the budget set and the budget line when \( p_1, p_2 \) and \( m \) are all multiplied by \( s > 0 \)?

(c) What happens to uncompensated demand when \( p_1, p_2 \) and \( m \) are all multiplied by \( s > 0 \)?

(d) How does utility change when \( p_1, p_2 \) and \( m \) are all multiplied by \( s > 0 \)?

(e) Suppose that the annual inflation rate is 50\%, the prices of all goods increase by 50\% and your income increases by 50\%. Does the inflation make you better off or worse off? Is real world inflation like this?

**Activity 4.3** The indirect utility function from the Cobb-Douglas utility function with \( a + b = 1 \) is \( u(x_1, x_2) = x_1^a x_2^b \) is:

\[
v(p_1, p_2, m) = \left( \frac{a^a b^b}{p_1^a p_2^b} \right) m.
\]

Confirm that this function is homogeneous of degree zero in \((p_1, p_2, m)\), increasing in \( m \) and decreasing in \( p_1 \).
4.7 Derivatives of homogeneous functions

Given the definition of a homogeneous function it is important to have notation for the derivatives of a function \( f \) at the point \((sx_1, sx_2, \ldots, sx_n)\). This is done by defining \( f_i(z_1, z_2, \ldots, z_n) \) by:

\[
f_i(z_1, z_2, \ldots, z_n) = \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_i}
\]

which is the derivative of \( f(z_1, z_2, \ldots, z_n) \) with respect to variable \( i \). This notation makes it possible to state and prove the following theorem.

**Theorem 11**

If the function \( f \) taking \( \mathbb{R}^{n+} \) into \( \mathbb{R} \) is homogeneous of degree \( s \) and differentiable then for any \((x_1, x_2, \ldots, x_n)\) in \( \mathbb{R}^{n+} \) and any \( s > 0 \):

\[
f_i(sx_1, sx_2, \ldots, sx_n) = s^{d-1}f_i(x_1, x_2, \ldots, x_n).
\]

Thus the partial derivative \( f_i \) is homogeneous of degree \( d-1 \).

**Proof.** If you are very comfortable with partial derivatives you can derive this result immediately by differentiating both sides of the equation:

\[
f(sx_1, sx_2, \ldots, sx_n) = s^d f(x_1, x_2, \ldots, x_n)
\]

with respect to \( x_i \) getting:

\[
s f_i(sx_1, sx_2, \ldots, sx_n) = s^d f_i(x_1, x_2, \ldots, x_n)
\]

and dividing by \( s \). If you need to take things more slowly define a function \( \phi \) by:

\[
\phi(s, x_1, x_2, \ldots, x_n) = f(sx_1, sx_2, \ldots, sx_n)
\]

so \( \phi(1, x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) \) and the condition:

\[
f(sx_1, sx_2, \ldots, sx_n) = s^d f(x_1, x_2, \ldots, x_n)
\]

that \( f \) is homogeneous of degree \( d \) becomes:

\[
\phi(s, x_1, x_2, \ldots, x_n) = s^d \phi(1, x_1, x_2, \ldots, x_n).
\]

Differentiating both sides of this equation with respect to \( x_i \) gives:

\[
\frac{\partial \phi(s, x_1, x_2, \ldots, x_n)}{\partial x_i} = s^d \frac{\partial \phi(1, x_1, x_2, \ldots, x_n)}{\partial x_i}.
\]

Now \( \phi(s, x_1, x_2, \ldots, x_n) = f(z_1, z_2, \ldots, z_n) \) where \( z_i = sx_i \) for \( i = 1, \ldots, n \), so using the chain rule:

\[
\frac{\partial \phi(s, x_1, x_2, \ldots, x_n)}{\partial x_i} = \frac{\partial z_i}{\partial x_i} \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_i} = s \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_i}.
\]
If \( s = 1 \) so \( z_i = x_i \) this equation becomes:

\[
\frac{\partial \phi(1, x_1, x_2, \ldots, x_n)}{\partial x_i} = \frac{\partial f(1, x_1, x_2, \ldots, x_n)}{\partial x_i}.
\]

The last three equations imply that:

\[
s \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_i} = s^d \frac{\partial f(1, x_1, x_2, \ldots, x_n)}{\partial x_i}
\]

so dividing by \( s \):

\[
\frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_i} = s^{d-1} \frac{\partial f(1, x_1, x_2, \ldots, x_n)}{\partial x_i}
\]

where \( z_i = sx_i \) for \( i = 1, \ldots, n \). Using notation \( f_i(z_1, z_2, \ldots, z_n) \) for \((\partial f(z_1, z_2, \ldots, z_n))/(\partial z_i)\) this becomes:

\[
f_i(sx_1, sx_2, \ldots, sx_n) = s^{d-1} f_i(x_1, x_2, \ldots, x_n).
\]

\[\blacksquare\]

**Activity 4.4** The Cobb-Douglas utility function \( u(x_1, x_2) = x_1^a x_2^b \) with \( a + b = 1 \) is homogeneous of degree one. Find the partial derivatives of the Cobb-Douglas utility function \( u(x_1, x_2) = x_1^a x_2^b \) where \( a + b = 1 \) and confirm that the function and its derivatives satisfy Theorem 11.

### 4.8 Homogeneous utility functions

#### 4.8.1 Homogeneous utility functions and the indifference curve diagram

If the utility function is homogeneous of degree \( d \) and \( u(x) = u_0 \) so \( x \) lies on the indifference curve with utility \( u_0 \), homogeneity implies that \( u(sx) = s^d u(x) = s^d u_0 \), so \( sx \) lies on the indifference curve with utility \( s^d u_0 \). Thus given an indifference curve with utility \( u_0 \) the points with utility \( u_1 \) can be found by multiplying the points on the \( u_0 \) indifference curve by \( s \) where \( s^d = u_1/u_0 \). In particular when \( d = 1 \) so the utility function is homogeneous of degree one, the indifference curve with utility \( u_1 \) can be found by treating the points on the \( u_0 \) indifference curve as vectors and multiplying them by \( u_1/u_0 \).

This is illustrated in Figure 4.1 which shows indifference curves with the utility function \( u(x_1, x_2) = x_1^{2/3} x_2^{1/3} \) which is homogeneous of degree one in \( x_1 \) and \( x_2 \). Given a point on the indifference curve \( x_1^{2/3} x_2^{1/3} = 1 \) it is possible to find the corresponding point on the indifference curve \( x_1^{2/3} x_2^{1/3} = 3 \) by multiplying the vector \((x_1, x_2)\) by 3. This point can be found geometrically by drawing a ray through the original point with utility 1, and finding the point \( s \) on the ray that is three times as far from the origin.
4. Homogeneous and homothetic functions in consumer choice theory

Figure 4.1: Indifference curves with a homogeneous utility function \( u(x_1, x_2) = \frac{x_1^{2/3}}{x_2^{1/3}} \).

4.8.2 Homogeneous utility functions and the marginal rate of substitution

Figure 4.1 shows the lines that are tangent to the indifference curves at points on the same ray. In the figure it looks as if lines on the same ray have the same gradient. This is indeed the case. The gradient of the tangent line is \(-MRS\) where \( MRS \) is the marginal rate of substitution, given by:

\[
MRS = \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)}
\]

where \( u_i(x_1, x_2) \) is marginal utility, that is the derivative:

\[
\frac{\partial u(x_1, x_2)}{\partial x_i}.
\]

Using notation \( u_i(sx_1, sx_2) \) for \((\partial u(z_1, z_2))/(\partial z_i)\) where \( z_i = sx_i \) gives the result:

\[
\frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{u_1(sx_1, sx_2)}{u_2(sx_1, sx_2)}
\]

that is the marginal rate of substitution is homogeneous of degree zero in \((x_1, x_2)\).

---

**Theorem 12**

If \( u \) is a homogeneous function from \( \mathbb{R}^{2+} \) into \( \mathbb{R} \) so:

\[
u(sx_1, sx_2) = s^d u(x_1, x_2)
\]

and \( u_2(x_1, x_2) \neq 0 \), then \( u_2(sx_1, sx_2) \neq 0 \) and:

\[
\frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{u_1(sx_1, sx_2)}{u_2(sx_1, sx_2)}
\]

that is the marginal rate of substitution is homogeneous of degree zero in \((x_1, x_2)\).
4.8. Homogeneous utility functions

**Proof.** From Theorem 11 if \( u(x_1, x_2) \) is homogeneous of degree \( d \) in \((x_1, x_2)\) with continuous partial derivatives then its partial derivatives are homogeneous of degree \( d - 1 \), so:

\[
\begin{align*}
    u_1(sx_1, sx_2) &= s^{d-1}u_1(x_1, x_2) \\
    u_2(sx_1, sx_2) &= s^{d-1}u_2(x_1, x_2)
\end{align*}
\]

so as \( u_2(x_1, x_2) \neq 0, u_2(sx_1, sx_2) \neq 0 \) and:

\[
\frac{u_1(sx_1, sx_2)}{u_2(sx_1, sx_2)} = \frac{s^{d-1}u_1(x_1, x_2)}{s^{d-1}u_2(x_1, x_2)} = \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)}.
\]

The term on the left is the MRS at the point \((sx_1, sx_2)\), the term on the right is the MRS at the points \((x_1, x_2)\).

\[
\blacksquare
\]

4.8.3 Homogeneous utility functions and uncompensated demand

Intermediate microeconomics uses the indifference curve diagram to show that the solution to the consumer’s problem is at a point where the budget line is tangent to the indifference curve, so the marginal rate of substitution is equal to the price ratio. As Figure 4.1 shows with a homogeneous utility function the marginal rate of substitution is the same at all points along a ray, so if income changes but prices do not change uncompensated demand moves along the ray. Thus if income doubles the demand for each good doubles but the ratio of consumption of good 2 to consumption of good 1 does not change. More generally any change in income affects the quantity of goods being consumed. This is a very strong condition; it says for example that if income doubles then both consumption of salt, and consumption of housing space double; whereas in fact as people get richer their pattern of consumption changes. In particular homogeneous utility functions imply that there are no inferior goods.

In fact this result on the pattern of consumption does not rely on the tangency condition, as the next result shows.

**Theorem 13**

If the utility function is homogeneous then for all prices \((p_1, p_2, \ldots, p_n)\) and income \(m > 0\):

\[
x_i(p_1, p_2, \ldots, p_n, m) = mx_i(p_1, p_2, \ldots, p_n, 1) \text{ for } i = 1, 2, \ldots, n
\]

or in vector notation:

\[
x(p, m) = mx(p, 1).
\]

In words this says that to find the consumption of goods at any level of income \(m\), find the consumption when \(m = 1\) and then multiply by \(m\).

**Proof.** Suppose the result is not true, that is there are prices \(p\) and income \(m\), for which \(x(p, m) \neq mx(p, 1)\), that is \(mx(p, 1)\) does not solve the utility maximising problem with income \(m\). As \(x(p, 1)\) solves the problem with income 1, it must satisfy
the budget constraint with income 1, so:
\[ p_1 x_1(p, 1) + p_2 x_2(p, 1) + \cdots + p_n x_n(p, 1) \leq 1 \]
and be in \( \mathbb{R}^2^+ \), so \( x_i(p, 1) \geq 0 \) for \( i = 1, \ldots, n \). This implies that if \( m > 0 \):
\[ p_1(m x_1(p, 1)) + p_2(m x_2(p, 1)) + \cdots + p_n(m x_n(p, 1)) \leq m \]
and \( m x_i(p, 1) \geq 0 \). Thus \( m x(p, 1) \) lies in the budget set with prices \( p \) and income \( m \). As \( m x(p, 1) \) does not solve the utility maximising problem but \( x(p, m) \) does:
\[ u(x(p, m)) > u(m x(p, 1)). \quad (4.3) \]
Because \( u \) is homogeneous of degree \( d \):
\[ u(x(p, m)) = u(mm^{-1}x(p, m)) = m^d u(m^{-1}x(p, m)) \]
and:
\[ u(m x(p, 1)) = m^d u(x(p, 1)) \]
so from (4.3):
\[ m^d u(m^{-1}x(p, m)) = u(x(p, m)) > u(m x(p, 1)) = m^d u(x(p, 1)) \]
so dividing by \( m^d \):
\[ u(m^{-1}x(p, m)) > u(x(p, 1)). \]
But this is impossible because as \( x(p, m) \) solves the problem with income \( m \), \( x(p, m) \) is feasible with income \( m \) implying that \( m^{-1}x(p, m) \) is feasible with income 1, so as \( x(p, 1) \) solves the problem with income 1:
\[ u(x(p, 1)) \geq u(m^{-1}x(p, m)). \]
Thus the assumption that the result does not hold leads to contradiction, so the result must hold. ■

**Activity 4.5** Consider the utility function \( x_1^a x_2^b \) where \( a > 0, b > 0 \) and \( a + b = 1 \). From Activity 4.4 this function is homogeneous of degree one.

(a) Find the MRS and show that it is homogeneous of degree zero in \( (x_1, x_2) \).

(b) Show that:
\[
x_1(p_1, p_2, m) = m x_1(p_1, p_2, 1) \\
x_2(p_1, p_2, m) = m x_2(p_1, p_2, 1).
\]

### 4.9 Homothetic functions

#### 4.9.1 Homogeneity and homotheticity

Homotheticity is a weaker condition than homogeneity.
4.9. Homothetic functions

A function $f$ from $\mathbb{R}^n$ into $\mathbb{R}$ is homothetic if it has the form:

$$f(x) = q(r(x))$$

where $r$ is a function that is homogeneous of degree one and $q$ is a strictly increasing function.

**Definition 22**

Any homogeneous function of degree $d > 0$ is homothetic.

**Proposition 9**

Any homogeneous function of degree $d > 0$ is homothetic.

**Proof.** If $f(x_1, x_2, \ldots, x_n)$ is homogeneous of degree $d > 0$ so for any $s > 0$:

$$f(sx_1, sx_2, \ldots, sx_n) = s^d f(x_1, x_2, \ldots, x_n)$$

let:

$$r(x_1, x_2, \ldots, x_n) = (f(x_1, x_2, \ldots, x_n))^{1/d}$$

which implies that:

$$r(sx_1, sx_2, \ldots, sx_n) = (f(sx_1, sx_2, \ldots, sx_n))^{1/d} = (s^d f(x_1, x_2, \ldots, x_n))^{1/d} = s (f(x_1, x_2, \ldots, x_n))^{1/d} = sr(x_1, x_2, \ldots, x_n)$$

so $r(x_1, x_2, \ldots, x_n)$ is homogeneous of degree one and:

$$f(x_1, x_2, \ldots, x_n) = q(r(x_1, x_2, \ldots, x_n))$$

where $q(r) = r^d$ so $q$ is a strictly increasing function of $r$. ■

### 4.9.2 Indifference curves with homothetic utility functions

Indifference curves are the level sets of the utility function, so an indifference curve is a set of the form:

$$\{(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2, u(x_1, x_2) = u_0\}.$$  

Now suppose that the utility function $u(x_1, x_2)$ is homothetic, so:

$$u(x_1, x_2) = q(r(x_1, x_2))$$

where $q$ is a strictly increasing function and $r$ is a function that is homogeneous of degree one. Thus $u(x_1, x_2) = u(x'_1, x'_2)$ implies that $q(r(x_1, x_2)) = q(r(x'_1, x'_2))$ which as $q$ is strictly increasing requires that $r(x_1, x_2) = r(x'_1, x'_2)$, so if $u_0 = q(r_0)$ the sets:

$$\{(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2, u(x_1, x_2) = u_0\}$$

$$\{(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2, r(x_1, x_2) = r_0\}$$
are the same. Thus any indifference curves associated with the homothetic utility function \( u \) is also a level set of the homogeneous function \( r(x_1, x_2) \). Diagrams of the indifference curves and level sets for homothetic and homogeneous utility functions look the same, although the number associated with each indifference curve is different for the two functions \( u \) and \( r \).

### 4.9.3 Marginal rates of substitution with homothetic utility functions

The fact that the indifference curves for homothetic and homogeneous indifference curves are the same implies that the geometric properties of the two diagrams are the same. In particular homothetic utility functions should share with homogeneous utility functions the property that moving along a ray the marginal rate of substitution does not change. This is indeed so. The formal result is:

**Theorem 14**

Suppose \( u \) is a homothetic function from \( \mathbb{R}^2^+ \) into \( \mathbb{R} \), so:

\[
 u(x_1, x_2) = 1(r(x_1, x_2))
\]

where \( q \) is strictly increasing, and \( r \) is homogeneous of degree one. Suppose also that the functions \( q \) and \( r \) are differentiable, and that for all \( (x_1, x_2) \) in \( \mathbb{R}^2^+ \), \( q'(r(x_1, x_2)) > 0 \) and \( r_2(x_1, x_2) \neq 0 \). Then \( u_2(x_1, x_2) \neq 0 \), \( u_2(sx_1, sx_2) \neq 0 \) and:

\[
 \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{u_1(sx_1, sx_2)}{u_2(sx_1, sx_2)}
\]

that is the marginal rate of substitution is homogeneous of degree zero in \( (x_1, x_2) \).

**Proof.** as \( u(x_1, x_2) = q(r(x_1, x_2)) \) the chain rule implies that:

\[
 u_1(x_1, x_2) = q'(r(x_1, x_2))r_1(x_1, x_2)
\]

\[
 u_2(x_1, x_2) = q'(r(x_1, x_2))r_2(x_1, x_2).
\]

From the assumptions of \( q' \) and \( r_2 \) the terms in the second equation are not zero and division gives:

\[
 \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{q'(r(x_1, x_2))r_1(x_1, x_2)}{q'(r(x_1, x_2))r_2(x_1, x_2)} = \frac{r_1(x_1, x_2)}{r_2(x_1, x_2)}.
\]

This argument applies to all points in \( \mathbb{R}^2^+ \). In particular it applies to \( (sx_1, sx_2) \) so:

\[
 \frac{u_1(sx_1, sx_2)}{u_2(sx_1, sx_2)} = \frac{r_1(sx_1, sx_2)}{r_2(sx_1, sx_2)}.
\]

As \( r(x_1, x_2) \) is homogeneous of degree one Theorem 11 implies that the derivatives of \( r \) are homogeneous of degree zero so:

\[
 r_1(x_1, x_2) = r_1(sx_1, sx_2)
\]

\[
 r_2(x_1, x_2) = r_2(sx_1, sx_2)
\]
so from (4.6) and (4.7):
\[
\frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{u_1(sx_1, sx_2)}{u_2(sx_1, sx_2)}.
\]

\[\blacksquare\]

4.9.4 Homothetic utility and uncompensated demand

Theorem 13 says that if the utility function is homogeneous then for all prices \((p_1, p_2, \ldots, p_n)\) and income \(m > 0\):
\[
x_i(p_1, p_2, \ldots, p_n, m) = mx_i(p_1, p_2, \ldots, p_n, 1) \quad \text{for } i = 1, 2, \ldots, n
\]
or in vector notation:
\[
x(p, m) = mx(p, 1).
\]
A similar result applies to homothetic functions.

**Theorem 15**

If the utility function is homothetic then for all prices \((p_1, p_2, \ldots, p_n)\) and income \(m > 0\):
\[
x_i(p_1, p_2, \ldots, p_n, m) = mx_i(p_1, p_2, \ldots, p_n, 1) \quad \text{for } i = 1, 2, \ldots, n
\]
or in vector notation:
\[
x(p, m) = mx(p, 1).
\]

**Proof.** Given the definition of a homothetic function if the utility function \(u(x)\) is homothetic \(u(x) = q(r(x))\) where \(q\) is a strictly increasing function and \(r\) is homogeneous of degree one. Then from Theorem 7 \(r(x)\) is also a utility function representing the same preferences as \(u(x)\). From Theorem 9 the solutions to the utility maximising problem with utility function \(u(x)\) are the same as the solutions to the utility maximising problem with utility function \(r(x)\). As \(r(x)\) is homogeneous Theorem 13 implies that the solutions satisfy:
\[
x(p, m) = mx(p, 1).
\]
Thus uncompensated demand with both homogeneous and homothetic utility functions has the property that the proportions in which goods are consumed do not change as income increases with prices remaining the same. \[\blacksquare\]

**Activity 4.6** For each of the following functions state whether or not it is homothetic in \((x_1, x_2)\).

(a) \(cx_1^3 + dx_1^2 x_2 + ex_1 x_2^2 + fx_1^3\)

(b) \((cx_1^3 + dx_1^2 x_2 + ex_1 x_2^2 + fx_1^3)/(ax_1 + bx_2)^3\)

(c) \(x_1^\alpha x_2^\beta\) where \(\alpha > 0\) and \(\beta > 0\), do not assume that \(\alpha + \beta = 1\)

(d) \(\alpha \ln x_1 + \beta \ln x_2\) where \(\alpha > 0\), \(\beta > 0\).

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Activity 4.7  Find the marginal rate of substitution for the utility function \( u(x_1, x_2) = (x_1 + 1)^a(x_2 + 1)^b \). Is this utility function homothetic?

4.10 Solutions to activities

Solution to Activity 4.1

(a) \( a(sx_1) + b(sx_2) = s(sx_1 + bx_2) \) so the function \( ax_1 + bx_2 \) is homogeneous of degree one.

(b) \( (a(sx_1) + b(sx_2))^2 = s^2(ax_1 + bx_2)^2 \) so the function \( (ax_1 + bx_2)^2 \) is homogeneous of degree 2.

(c) \( a(sx_1^2) + bsx_2 = s^2ax_1^2 + sbx_2 \), the function \( ax_1^2 + bx_2 \) is not homogeneous unless \( a = 0 \) or \( b = 0 \).

To see why, consider the case where \( a = b = 1 \), so \( f(x_1, x_2) = x_1^2 + x_2 \), \( f(1, 1) = 2 \), \( f(2, 2) = 6 \) and \( f(4, 4) = 20 \). If the function were homogeneous of degree \( d \) then:

\[
6 = f(2, 2) = 2^df(1, 1) = 2^d(2) \\
20 = f(4, 4) = 4^df(1, 1) = (4^d)(2).
\]

The first of these equations implies that \( 4^d = 2^d = 9 \), the second that \( 4^d = 10 \), so there is no \( d \) that satisfies both of these equations.

(d)

\[
c(sx_1^3) + d(sx_1)^2(sx_2) + e(sx_1)(sx_2)^2 + f(sx_2)^3 \\
= s^3(cx_1^3 + ex_1 x_2^2 + ex_1^2 x_2^2 + f x_1^3)
\]

so the function \( cx_1^3 + ex_1 x_2^2 + ex_1^2 x_2^2 + f x_1^3 \) is homogeneous of degree 3.

(e)

\[
c(sx_1^3) + d(sx_1)^2(sx_2) + e(sx_1)(sx_2)^2 + f(sx_2)^3 \\
= s^3(cx_1^3 + dx_1^2 x_2 + ex_1 x_2^2 + f x_1^3) \\
= cx_1^3 + dx_1^2 x_2 + ex_1 x_2^2 + f x_1^3 \\
= (ax_1 + bx_2)^3
\]

so the function:

\[
\frac{cx_1^3 + dx_1^2 x_2 + ex_1 x_2^2 + f x_1^3}{(ax_1 + bx_2)^3}
\]

is homogeneous of degree zero.

(f) \( (sx_1)^\alpha(sx_2)^\beta = s^{\alpha+\beta}(x_1^\alpha x_2^\beta) \) so the function \( x_1^\alpha x_2^\beta \) is homogeneous of degree \( \alpha + \beta \).

(g)

\[
\alpha \ln(sx_1) + \beta \ln(sx_2) = \alpha \ln x_1 + \beta \ln x_2 + (\alpha + \beta) \ln s
\]

the function \( \alpha \ln x_1 + \beta \ln x_2 \) is not homogeneous.
Solution to Activity 4.2

For any $s > 0$ the values of $(x_1, x_2)$ that satisfy the budget constraint $p_1x_1 + p_2x_2 \leq m$ are the same as the values that satisfy that constraint $(sp_1)x_1 + (sp_2)x_2 \leq (sm)$. In particular this applies when $s = 2$. The non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ are unchanged by changes in prices and income. Thus the budget line and budget set do not change when $(p_1, p_2, m)$ is replaced by $(sp_1, sp_2, sm)$, so the point $(x_1, x_2)$ that maximises $u(x_1, x_2)$ on the budget set does not change, implying that:

\[
x_1(p_1, p_2, m) = x_1(sp_1, sp_2, sm)
\]
\[
x_2(p_1, p_2, m) = x_2(sp_1, sp_2, sm)
\]

and:

\[
v(p_1, p_2, m) = u(x_1(p_1, p_2, m), x_2(p_1, p_2, m))
\]
\[
= u(x_1(sp_1, sp_2, sm), x_2(sp_1, sp_2, sm))
\]
\[
= v(sp_1, sp_2, sm)
\]

so $v(p_1, p_2, m) = v(sp_1, sp_2, sm)$. Thus uncompensated demand and the indirect utility function are homogeneous of degree zero in $(p_1, p_2, m)$.

Under the assumptions of this model if all prices and income increase by 50% it has no effect on utility. Real world inflation is not like this, the prices of different goods and the incomes of different people change by different percentages so some people will be better off and others worse off. In addition high rates of inflation are associated with considerable uncertainty which causes additional problems.

Solution to Activity 4.3

For the Cobb-Douglas utility function:

\[
v(p_1, p_2, m) = \left( \frac{a^ap^b}{p_1^ap_2^b} \right) m
\]

so:

\[
v(sp_1, sp_2, sm) = \left( \frac{a^ap^b}{(sp_1)^a(sp_2)^b} \right) (sm)
\]
\[
= s \frac{s^a+b}{s^{a+b}} \left( \frac{a^ap^b}{p_1^ap_2^b} \right) m = \left( \frac{a^ap^b}{p_1^ap_2^b} \right) m = v(p_1, p_2, m)
\]

so $v(p_1, p_2, m)$ is homogeneous of degree zero in $(p_1, p_2, m)$. Also:

\[
\frac{\partial v(p_1, p_2, m)}{\partial m} = \left( \frac{a^ap^b}{p_1^ap_2^b} \right) > 0
\]
\[
\frac{\partial v(p_1, p_2, m)}{\partial p_1} = -ap_1^{-a-1} \left( \frac{a^ap^b}{p_2^b} \right) m < 0
\]

so $v(p_1, p_2, m)$ is increasing in $m$ and decreasing in $p_1$. 

4. Homogeneous and homothetic functions in consumer choice theory

Solution to Activity 4.4

As \( a + b = 1 \) the partial derivatives of \( u(x_1, x_2) = x_1^a x_2^b \) are:

\[
\begin{align*}
\frac{\partial u}{\partial x_1} &= a x_1^{a-1} x_2 = a x_1^{-b} x_2^b = a \left( \frac{x_2}{x_1} \right)^b \\
\frac{\partial u}{\partial x_2} &= b x_2^{a-1} x_1^b = b x_2^{-a} x_1^a = b \left( \frac{x_1}{x_2} \right)^a.
\end{align*}
\]

The function is homogeneous of degree one, the derivatives are homogeneous of degree zero.

Solution to Activity 4.5

(a) The MRS for the Cobb-Douglas utility function \( u(x_1, x_2) = x_1^a x_2^b \) is:

\[
\frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} = \frac{a x_1^{a-1} x_2}{b x_2^{a-1} x_1} = \left( \frac{a}{b} \right) \left( \frac{x_2}{x_1} \right)
\]

which is homogeneous of degree zero in \((x_1, x_2)\) because \(sx_1/sx_2 = x_1/x_2\).

(b) For a Cobb-Douglas utility function:

\[
\begin{align*}
x_1(p_1, p_2, m) &= \frac{a}{p_1} m = m \left( \frac{a}{p_1} \right) = mx_1(p_1, p_2, 1) \\
x_2(p_1, p_2, m) &= \frac{b}{p_2} m = m \left( \frac{b}{p_2} \right) = mx_2(p_1, p_2, 1).
\end{align*}
\]

Solution to Activity 4.6

(a) The function \( cx_1^3 + ex_1^2 x_2 + ex_1 x_2^2 + fx_2^3 \) is homogeneous of degree \( 3 > 0 \) so from Proposition 9 is homothetic.

(b) The function:

\[
\frac{cx_1^3 + dx_1^2 x_2 + ex_1 x_2^2 + fx_2^3}{(ax_1 + bx_2)^3}
\]

is homogeneous of degree zero. It is not homothetic because no functions of degree zero are homothetic.

To see why this is, note that homotheticity implies that if \( s > 1 \):

\[
\begin{align*}
f(sx_1, sx_2, \ldots, sx_n) &= q(r(sx_1, sx_2, \ldots, sx_n)) \\
&= q(rs(x_1, x_2, \ldots, x_n)) \\
&> q(r(x_1, x_2, \ldots, x_n)) = f(x_1, x_2, \ldots, x_n)
\end{align*}
\]

because the function \( q \) is strictly increasing, whereas homogeneity of degree zero implies that:

\[
f(sx_1, sx_2, \ldots, sx_n) = f(x_1, x_2, \ldots, x_n).
\]
(c) The function \( \alpha \ln x_1 + \beta \ln x_2 \) where \( \alpha > 0, \beta > 0 \) is homothetic because:

\[
\alpha \ln x_1 + \beta \ln x_2 = \ln x_1^\alpha + \ln x_2^\beta \\
= \ln \left( x_1^\alpha x_2^\beta \right) \\
= \ln \left( \left( \frac{x_1^\alpha}{x_2^{1+\beta}} \right) \left( \frac{x_2^{1+\beta}}{x_1^\alpha} \right) \right) \\
= (\alpha + \beta) \ln \left( \frac{x_1^{\alpha+\beta}}{x_2^{\alpha+\beta}} \right).
\]

The function:

\[
\frac{x_1^\alpha}{x_2^{\alpha+\beta}} \frac{x_2^\beta}{x_1^{\alpha+\beta}}
\]

is homogeneous of degree one, the function \( (\alpha + \beta) \ln r \) is strictly increasing so \( \alpha \ln x_1 + \beta \ln x_2 \) is homothetic.

**Solution to Activity 4.7**

With the utility function \( u(x_1, x_2) = (x_1 + 1)^a(x_2 + 1)^b \):

\[
MRS = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} = \frac{a(x_1 + 1)^{a-1}(x_2 + 1)^b}{b(x_1 + 1)^a(x_2 + 1)^{b-1}} \\
= \frac{a(x_2 + 1)}{b(x_1 + 1)}.
\]

From Theorem 14 if the function \( u(x_1, x_2) = (x_1 + 1)^a(x_2 + 1)^b \) were homothetic the \( MRS \) would be homogeneous of degree zero in \( (x_1, x_2) \), which it is not. For example at \( (1, 2) \) the \( MRS \) is \( 2a/3b \), whereas at \( (2, 4) \) the \( MRS \) is \( 3a/4b \).

**A reminder of your learning outcomes**

By the end of this chapter, you should be able to:

- state the definitions of homogeneous and homothetic functions
- explain why the uncompensated demand and indirect utility functions are homogeneous of degree zero in prices and income
- determine when the indirect utility function is increasing or non-decreasing in income and decreasing or non-increasing in prices
- explain the relationship between the homogeneity of a function and its derivative
- describe the implications of homogeneous and homothetic utility functions for the shape of indifference curves, the marginal rate of substitution and uncompensated demand.
4.11 Sample examination questions

Note that many of the activities of this chapter are also examples of examination questions.

1. (a) Define the terms homogeneous and homothetic.
   (b) Are all homogeneous functions homothetic?
   (c) Are all homothetic functions homogeneous?
   (d) Suppose a consumer has a homogeneous utility function, income doubles, but prices do not change. What happens to uncompensated demand?
   (e) Suppose a consumer has a homothetic utility function, income does not change, but prices fall by 50%. What happens to uncompensated demand?

2. Which of the following statements are true? Explain your answers.
   (a) Uncompensated demand is homogeneous of degree zero in prices and income.
   (b) The indirect utility function is homogeneous of degree one in prices and income.
   (c) All utility functions are homogeneous.
   (d) All homogeneous utility functions are homothetic.