Management mathematics
R.D. Hewins
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Undergraduate study in
Economics, Management,
Finance and the Social Sciences

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Preface

0.1 Introduction

People in business, economics and the social sciences are increasingly aware of the need to be able to handle a range of mathematical tools. This course is designed to fill this need by extending the 100 courses in Mathematics and Statistics into several even more practical and powerful areas of mathematics. It is not just forecasting and index numbers that have uses. Such things as differential equations and stochastic processes, for example, do have direct, frequent and practical applications to everyday management situations.

This course is intended to extend your mathematical ability and interests beyond the knowledge acquired in earlier 100 courses. Throughout the mathematical and quantitative courses of the degrees we attempt to emphasise the applications of mathematics for management problems and decision-making. MT2076 Management mathematics is no exception. However, you must always recognise the need to ‘walk before you can run’ and hence new topics sometimes need to be covered in a relatively detailed mathematical way before the topics’ uses can be emphasised by more interesting and practical examples. It must be admitted that many good managers are not very mathematically adept. However, they would be even more inquisitive, more precise, more accurate in their statements, more selective in their use of data, more critical of advice given to them, etc. if they had a better grasp of quantitative subjects. Mathematics is an important tool which all good managers should appreciate.

Many of the topics within this course are extensions of the comparatively simple ideas covered within your 100 courses in Mathematics and Statistics. Other topics are fundamentally new. The course therefore both extends and reinforces existing knowledge and introduces new areas of interest and applications of mathematics in the ever-widening field of management.

0.2 Aims

- To extend your mathematical and statistical ability and interests beyond the knowledge acquired in your 100 courses in Mathematics and Statistics.

- To introduce new areas of interest and applications of mathematics and statistics in the ever-widening field of management.

- To familiarise yourself with, and become competent in, dynamic models and multivariate (as well as univariate) data analysis.
0.3 Learning outcomes

By the end of this course and having completed the Essential reading and activities, you should:

- be able to demonstrate further mathematical and statistical knowledge
- be able to apply mathematics at varying levels to aid decision-making
- understand how to analyse complex multivariate data sets with the aim of extracting the important message contained within the huge amount of data which is often available
- be able to construct appropriate models and interpret the results generated (this will often be a case of understanding the output from a computerised model)
- be able to demonstrate the wide applicability of mathematical models while, at the same time, identifying their limitations and possible misuse
- discuss the more technical/mathematical/theoretical side of management (including finance and economics)
- be able to read management/financial journals in the areas of (for example) management science and operations research, forecasting, financial engineering and market analysis, economics and econometrics and so on with a reasonably high level of assessing of the basic mathematical techniques employed therein.

0.4 Syllabus

- Logical use of set theory and Venn diagrams.
- Index numbers.
- Trigonometric functions. Imaginary numbers. (The prime requirement for both these topics is for modelling of cyclical dynamics via difference and differential equations).
- Difference (first and second order) and differential equations (linear, first and second order). Simultaneous second order equations.
- Simple stochastic processes – including ‘Gamblers ruin’, ‘Birth and Death’ and queuing models. Analysis of queues to include expected waiting time and expected queue length.
- Time series analysis. Forecasting techniques (including exponential smoothing, moving averages, trend and seasonality, simple Box-Jenkins (ARIMA)).

- Introduction to econometrics. Multiple regression (including using \( F \) tests). Simple analysis of variance.

- Principles of mathematical modelling.

- Clustering techniques and appreciation of other models. Data reduction models. Interpreting various types of scatter plots.

### 0.5 Overview of topics

The additional mathematical knowledge acquired by this course on top of your existing mathematical learning will enable you to:

- formulate and analyse managerial problems in a mathematical manner

- use mathematical logic (via Venn diagrams) to evaluate the sense, or otherwise, of given data

- understand the widely used (and sometimes misused) concept of index numbers

- apply difference and differential equations to models and solve dynamic relationships (for example, in sales, advertising, marketing, pricing, market analysis, financial markets, etc.)

- comprehend the uses of stochastic processes and Markov chains in predicting distributions for future outcomes

- analyse and model a series of observations over time (a ‘time series’) and forecast ahead (an obviously useful attribute of a good manager)

- understand the principles of mathematical models; ask sensible questions about assumptions, validity of results, etc.

- understand how econometric models can be used for analysis of complex economic relationships

- use relatively complex and powerful data reduction techniques in analysing the typically multidimensional observations available to a manager

- analyse multivariate data with a variety of techniques.

The above is a list of some of the specific knowledge which the successful student will acquire. They represent a summary of the main chapters of the guide. Only the topics of Chapter 3 are not specifically mentioned above as they are really a ‘means to an end’, i.e. a preliminary chapter on trigonometric functions and imaginary numbers which are required (among other things) for some of the subsequent chapters on difference and differential equation analysis. Nonetheless, there are occasionally examination questions specifically on Chapter 3 material.
Although containing many worked examples and, obviously, covering the whole syllabus, this subject guide is not intended as a textbook and should not be treated as such. However, if you use it correctly, it will provide a good indication of the levels required and typical areas of application. A full understanding and appreciation comes with practice, however, and, to this end, various texts are recommended for reading. Many of these texts also have worked examples and exercises which you should systematically work through.

0.6 Essential reading

In order to use this subject guide fully you should acquire a copy of the following essential textbooks, some of which tackle the majority of the course while others cover only specific parts:


A useful introductory textbook which is used in this course is:


Detailed reading references in this subject guide refer to the editions of the set textbooks listed above. New editions of one or more of these textbooks may have been published by the time you study this course. You can use a more recent edition of any of the books; use the detailed chapter and section headings and the index to identify relevant readings. Also check the virtual learning environment (VLE) regularly for updated guidance on readings.

0.7 Further reading

Please note that as long as you read the Essential reading you are then free to read around the subject area in any text, paper or online resource. You will need to support your learning by reading as widely as possible and by thinking about how these principles apply in the real world. To help you read extensively, you have free access to the VLE and University of London Online Library (see below).

The following books are a selection of additional texts covering certain aspects of the course. They vary considerably in level and coverage of material. You should establish whether they are necessary for you bearing in mind your own mathematical knowledge and abilities. You are strongly advised to check the nature of these books (via the internet for example) before contemplating purchasing any of them.
In addition, many of you will have used (and perhaps acquired) some of the mathematical/statistical texts for earlier courses. Where appropriate, reference is made to the relevant chapters of these books. They are indicated by an asterisk (*) in the list below.


0.8 Online study resources

In addition to the subject guide and the Essential reading, it is crucial that you take advantage of the study resources that are available online for this course, including the VLE and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at:

http://my.londoninternational.ac.uk

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged in to the Student Portal in order to register. As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you have forgotten these login details, please click on the ‘Forgotten your password’ link on the login page.

0.8.1 The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of subject material.

- Electronic study materials: The printed materials that you receive from the University of London are available to download, including updated reading lists and references.

- Past examination papers and Examiners’ commentaries: These provide advice on how each examination question might best be answered.
0.9. How to use the subject guide

- A student discussion forum: This is an open space for you to discuss interests and experiences, seek support from your peers, work collaboratively to solve problems and discuss subject material.

- Videos: There are recorded academic introductions to the subject, interviews and debates and, for some courses, audio-visual tutorials and conclusions.

- Recorded lectures: For some courses, where appropriate, the sessions from previous years’ Study Weekends have been recorded and made available.

- Study skills: Expert advice on preparing for examinations and developing your digital literacy skills.

- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

0.8.2 Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login:

http://tinyurl.com/ollathens

The easiest way to locate relevant content and journal articles in the Online Library is to use the Summon search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please see the online help pages:

www.external.shl.lon.ac.uk/summon/about.php

0.9 How to use the subject guide

The chapters of this subject guide follow a similar format and, unless indicated otherwise, you should tackle the study of each topic in the following way:

1. Read the relevant chapter of the subject guide. You may be referred back to earlier chapters if a refreshment of ideas is required.

2. Then do the reading from the essential textbooks.

3. Go through the worked examples and then tackle as many problems as possible yourself. Remember that learning mathematics is best done by attempting problems, not solely by reading.
In planning the workload associated with the course, you should appreciate that the chapters of this subject guide are of different lengths and will therefore take a different amount of time to cover. However, to help your time management the chapters and topics of the course are converted below into approximate weeks of a typical 30-week university course.

- Chapter 1: 2 weeks
- Chapter 2: 2 weeks
- Chapter 3: 2 weeks
- Chapter 4: 2 weeks
- Chapter 5: 3 weeks
- Chapter 6: 3 weeks
- Chapter 7: 3 weeks
- Chapter 8: 3 weeks
- Chapter 9: 3 weeks
- Chapter 10: 4 weeks
- Chapter 11: 2 weeks
- Chapter 12: 1 week

**TOTAL** 30 weeks

### 0.10 Examination advice

**Important:** the information and advice given here are based on the examination structure used at the time this guide was written. Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current Regulations for relevant information about the examination, and the VLE where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

The course is assessed by a three-hour unseen written examination. Candidates should answer all **eight** questions. Questions will often consist of several parts – part marks will be noted where appropriate on the examination paper. There will be a mixture of problem-solving and comment-based questions.

A Sample examination paper is provided at the end of this subject guide.

Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and *Examiners’ commentaries* for the course which give advice on how each question might best be answered.

### 0.11 Examination technique

Examinations are nerve-racking occasions. If you are particularly prone to examination nerves then it is a good idea to familiarise yourself with the examination situation by setting yourself three-hour time slots to do questions in time-constrained examination
conditions. Eventually you can use past papers as mock examinations – in the meantime create one for yourself from questions within a book. This is worthwhile doing!

Remember that even generous Examiners cannot award marks for blank pages! It is surprising how many students fail to answer enough questions, fail to write comments when required or fail to give sufficient explanation. All these failings are extremely noteworthy – make certain you avoid them.

In the Examiners’ marking scheme for quantitative subjects, marks are almost always awarded for method as well as accuracy. Bear this in mind when tackling problems.

State clearly any assumptions you feel it is necessary to make.

Become very familiar with how to operate your calculator. A calculator may be used when answering questions on this paper and it must comply in all respects with the specification given in the Regulations and on your Admissions Notice. The make and type of machine must be clearly stated on the front cover of the answer book.
Chapter 1
Set theory

1.1 Aims of the chapter

- To introduce the alternative common notations for set theory.
- To extend your knowledge of sets, beyond the basics which you will already have encountered, with a greater emphasis on interpretation and logic.
- To establish the usefulness of a diagrammatic approach to logic and data summary.

1.2 Learning outcomes

By the end of this chapter, and having completed the Essential reading and activities, you should be able to:

- understand the basic notation and nomenclature of sets especially: unions, intersections, complements, null sets, subsets, finite and infinite sets, differences of sets, the order of a set, the universal set
- construct a Venn diagram (not always straightforward!) from given relational data
- interpret Venn diagrams and set notation and explain their meaning in non-mathematical (‘everyday’) English
- use sets and Venn diagrams to analyse data (for example, to show inconsistencies and to derive maximum and/or minimum orders of sets).

1.3 Essential reading

Unfortunately, none of the essential textbooks covers this topic and, to the knowledge of the author, no text covers set theory in a similar manner to the way in which the topic is covered within this chapter. There are several illustrative examples within this chapter to show the particular application of sets to management situations, etc. In addition, you are particularly urged to review past papers.

1.4 Further reading

Some reference to sets is made in Anthony and Biggs (Chapter 2); Barnett and Zeigler (Appendix A1); Booth (Module 25).
1. Set theory

1.5 Introduction

Set theory is a relatively new aspect of mathematics which is now taught at all levels of education, from primary school upwards. The reason for this is its wide applicability in denoting and enumerating events and its visual appeal in using Venn diagrams. Although hidden in new nomenclature and notation, set theory is really only a combination of logic and enumeration of events. One important use is in probability theory but its elegance and efficiency in portraying logical associations demands every student’s attention. Within this course we concentrate on using sets for logic analysis.

There are numerous examples of set theory questions employed for logic analysis within past examination papers for MT2076 Management mathematics and you are strongly advised to use some of them at some stage of your preparation.

1.6 Sets

A set is simply a collection of things or objects, of any kind. These objects are called elements or members of the set. We refer to the set as an entity in its own right and often denote it by $A$, $B$, $C$ or $D$, etc.

If $A$ is a set and $x$ a member of the set, then we say $x \in A$, i.e. $x$ ‘belongs to’ $A$. The symbol $\notin$ denotes the negation of $\in$ i.e. $x \notin A$ means ‘$x$ does not belong to $A$’.

The elements of a set, and hence the set itself, are characterised by having one or more properties that distinguish the elements of the set from those not in the set, for example if $C$ is the set of non-negative real numbers, then we might use the notation

$$C = \{x \mid x \text{ is a real number and } x \geq 0\}$$

i.e. the set of all $x$ such that $x$ is a real number and non-negative.

| Example 1.1 | If $A$ is the set of all integers then $4 \in A$ but $6.7 \notin A$. |

Since sets are determined by their elements we say that $A = B$ if and only if they have the same elements.

$\emptyset$ represents the empty or null set, i.e. a set containing no elements.

The set containing everything is termed the universal set and is usually written as $U$.

1.7 Subsets

If $A$ and $B$ are two sets and all the elements of $A$ also belong to $B$ then it can be said that:

- $A$ is contained in $B$
- or $A$ is a subset of $B$
- or $B$ contains $A$.

These expressions are all equivalent and may be symbolically written as $A \subseteq B$. 

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1.8 The order of sets: finite and infinite sets

A set is said to be finite if it contains only a finite number of elements; otherwise the set is an infinite set. The number of elements in a set \( A \) is called the order of \( A \) and is denoted by \( |A| \) or \( n(A) \) or \( n_A \).

**Example 1.2** The set of all integers is an infinite set.

**Example 1.3** The set of days in a week has order 7.

1.9 Union and intersection of sets

The union of two sets \( A \) and \( B \) is a set containing all the elements in either \( A \) or \( B \) (or both)

\[
A \cup B = \{ x \mid x \in A \text{ or } x \in B \}.
\]

The intersection of two sets \( A \) and \( B \) is a set containing all the elements that are both in \( A \) and \( B \)

\[
A \cap B = \{ x \mid x \in A \text{ and } x \in B \}.
\]

If sets \( A \) and \( B \) have no elements in common, i.e. \( A \cap B = \emptyset \), then \( A \) and \( B \) are termed disjoint sets.

The above notation can be extended into the case of a family of sets (for example, \( A_i, i = 1, 2, \ldots, k \)). Thus the union of the family is

\[
\bigcup A_i = \{ x \mid x \in A_i \text{ for some } i = 1, 2, \ldots, k \}, \quad i = 1, 2, \ldots, k. \tag{1.1}
\]

The intersection of the family is:

\[
\bigcap A_i = \{ x \mid x \in A_i \text{ for every } i = 1, 2, \ldots, k \}, \quad i = 1, 2, \ldots, k. \tag{1.2}
\]

**Example 1.4** If \( A = \{1, 3, 5, 7\} \) and \( B = \{1, 2, 3, 4, 5\} \), then
\( A \cup B = \{1, 2, 3, 4, 5, 7\} \) and \( A \cap B = \{1, 3, 5\} \).

**Activity 1.1** If \( A = \{a, b, c, d, e, f\} \), \( B = \{a, e, g, h, j\} \) and \( C = \{b, c, f, g\} \), what are the following subsets?

(a) \( A \cup B \)
(b) \( B \cap C \)
(c) \( A \cap B^c \)
(d) \( A \cap (B \cup C) \).
1. Set theory

1.10 Differences and complements

If $A$ and $B$ are sets then the **difference** set $A - B$ is the set of all elements of $A$ which do not belong to $B$.

If $B$ is a subset of $A$, then $A - B$ is sometimes called the **complement** of $B$ in $A$.

When $A$ is the universal set one may simply refer to the complement of $B$ to denote all things not in $B$. The complement of a set $A$ is denoted as $A^c$ or $\bar{A}$.

**Note: De Morgan’s Theorems**

\[
(A \cap B)^c = A^c \cup B^c \\
(A \cup B)^c = A^c \cap B^c.
\]

The above relationships are most easily confirmed by using a Venn diagram (see below) to indicate that both sides of the above equations amount to the same areas of the diagram.

1.11 Venn diagrams

Often the relationships that exist between sets can best be shown using a Venn diagram. To construct a Venn diagram we let a certain region, usually a rectangle, represent the universal set. This rectangle is often implied by the constraints of the page and only in those circumstances where its boundary is important is the rectangle drawn (see the diagrams below for example). Individual sets are then represented by regions, often circles, within this rectangle. One can then easily depict intersections, unions, complements, etc. on the diagram. Figure 1.1 shows Venn diagram examples.

![Figure 1.1: Venn diagram examples.](image-url)
In Figure 1.1 the triple set Venn diagram is particularly useful and can be used to show, for example, that

\[ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C). \]

However, beware of trying to solve all problems from an equation point of view (involving perhaps unions, intersections and complements). Many problems are better tackled from a logical argument. See Examples 1.5 and 1.6 below.

**Activity 1.2** Construct Venn diagrams involving \( A, B \) and \( C \) to show each of the following subsets:

(a) \( A \cup (B \cap C^c) \)
(b) \( (A \cup B \cup C)^c \)
(c) \( B \cup A^c \)
(d) \( (A \cup B) \cap (B \cup C) \).

**Example 1.5** A publishing company has three main magazine publications \( A, B \) and \( C \). A market survey on the reading habits of 200 people surveyed revealed:

- 84 read magazine A
- 111 read magazine B
- 73 read magazine C
- 59 read \( A \) and \( B \)
- 53 read \( B \) and \( C \)
- 32 read \( A \) and \( C \)
- 20 read all three magazines.

How many of those people surveyed:

(a) Read just one of the magazines?
(b) Read just two of the magazines?
(c) Read none of the magazines?

This problem can be solved by putting the information into a Venn diagram, as shown in Figure 1.2.

The number of elements in each region might be calculated in a number of ways. Perhaps starting from the centre and working outwards is the best idea here. Since 20 people read \( A, B \) and \( C \) and 32 read \( A \) and \( C \) then 12 must read \( A \) and \( C \) but not \( B \), etc. Hence from the diagram we have the answers:

(a) \( 13 + 19 + 8 = 40 \)
1. Set theory

(b) $39 + 12 + 33 = 84$

(c) 56.

Figure 1.2: Venn diagram for Example 1.5.

Activity 1.3  (Slightly more difficult)

An insurance company insures 20,000 businesses against the perils of fire, flood and storm damage. During a 10-year period 99% of these businesses make no claim at all against the insurance company. No business claims for more than one type of peril at a time, but of those businesses that have made one or more claims during the stated 10-year period:

- 40% have claimed for fire damage
- 50% have claimed for flood damage
- 38% have claimed for storm damage
- 10% have claimed on different occasions for fire and storm damage
- 15% have claimed on different occasions for storm and flood damage
- 5% have claimed on different occasions for fire and flood damage.

(a) How many businesses have claimed for all three types of damages (fire, flood and storm) on separate occasions?

(b) Assuming no business has claimed for the same type of damage more than once, how many claims in total have been made?

Example 1.6  Of 30 employees in the marketing department of a large multinational company, 24 are male, 20 have university degrees and 22 have had experience in other companies.

(a) What is the fewest possible number of male degree holders with no experience in other companies that might be in the department?
(b) What is the greatest possible number of female employees in the department who do not have a university degree but have experience in other companies?

From the given information we know there are 6 females, 10 without university degrees and 8 with no experience in another company. The answer to (a) occurs when the females ‘use up’ as many ‘degrees’ and ‘no experiences’ as possible, i.e. 6 ‘degrees’ and 6 ‘no experiences’. Furthermore, we know that if \( M = \text{set of males}, \ D = \text{set of degree holders}, \ E = \text{set with experience in another company} \) then

\[
\begin{align*}
n(M \cup D) & \leq 30 \\
n(M \cup D) & = n(M) + n(D) - n(M \cap D)
\end{align*}
\]

and since

\[
n(M \cap D) \geq 24 + 20 - 30 = 14.
\]

Similarly, \( n(M \cap E) \geq 16 \) and \( n(D \cap E) \geq 12 \). To satisfy these conditions we might try setting \( n(M \cap D \cap E^c) = 0 \).

(a) The Venn diagram in Figure 1.3 seems to satisfy all the conditions and hence it is possible that there are no male degree holders with no experience in other companies.

(b) Using a similar logic, it is possible that there are 6 females satisfying the conditions (i.e. do not have a university degree, but have experience in other companies) as indicated in Figure 1.4.

\[\text{Figure 1.3: Venn diagram for Example 1.6.}\]

\[\text{Figure 1.4: Venn diagram for Example 1.6.}\]
1. Set theory

We have approached Example 1.6 in a sort of ‘trial and error’ approach. See Example 1.7 below and Section 1.16 for examples where we determine possible orders of sets in a more structured fashion.

**Example 1.7** If \( n(X \cup Y \cup Z) = 25; n(X \cap Y \cap Z) = 5; n(X \cap Y) = 8; n(Y \cap Z) = 9; n(X \cap Y^c \cap Z^c) = 2 \) and \( n(X) = n(Y) = n(Z) \), what is \( n(X) \)?

From the given information we can construct the following Venn diagram in Figure 1.5.

Letting \( x, y \) and \( z \) be the number of members in the unknown areas we have the following equations:

\[
(1): \quad 25 = 2 + 3 + 5 + x + y + 4 + z \quad \Rightarrow \quad x + y + z = 11.
\]

Furthermore since \( n(X) = n(Y) = n(Z) \) we have

\[
10 + x = 12 + y = 9 + x + z
\]

which means that \( z = 1 \) and \( x = y + 2 \). Substituting in (1), we have

\[
x + x - 2 + 1 = 11 \quad \Rightarrow \quad x = 6.
\]

Hence \( n(X) = 2 + 3 + 5 + 6 = 16 \).

![Venn diagram for Example 1.7](image)

**Figure 1.5:** Venn diagram for Example 1.7.

**Activity 1.4** Of 20 management trainees in a large company, 16 are male, 15 are graduates and 10 have had at least three years’ experience. Determine:

(a) the minimum number of males with at least three years’ experience
(b) the maximum number of female graduates who have had at least three years’ experience.

1.12 Logic analysis

Sets and Venn diagrams are particularly useful in depicting the interrelationships between sets and also analysing whether given information makes sense or not. These ideas are best illustrated by examples.
Example 1.8  A company studied the preferences of 10,000 of its customers for its products $A$, $B$ and $C$. They discovered that 5,010 liked product $A$, 3,470 liked product $B$ and 4,820 liked product $C$. All products were liked by 500 people, products $A$ and $B$ (and perhaps $C$) were liked by 1,000 people, products $A$ and $C$ (and perhaps $B$) were liked by 840 people and products $B$ and $C$ (and perhaps $A$) were liked by 1,410 people.

(a) Draw a Venn diagram to illustrate the above information and show that there must be an error in the data provided.

(b) If the erroneous data are for those people liking products $B$ and $C$ (and perhaps $A$) determine:
   i. its correct value if all 10,000 customers like at least one product
   ii. upper and lower limits on its value if some customers like none of the products.

Suggested solution:

(a) Construct the Venn diagram in Figure 1.6. It is often a good idea to start from the triple intersection and work outwards.
Total customers $= 3670 + 340 + 500 + 500 + 1560 + 910 + 3070 = 10550$. Hence, there must be an error in the data.

(b) Let $n(B \cap C) = x$, then the Venn diagram becomes that shown in Figure 1.7.
   i. If the total customers liking $A$, $B$ and $C = 10,000$ then

   
   $10000 = 3670 + 500 + 500 + 340 + 2970 - x + x - 500 + 4480 - x = 11960 - x.$

   Hence $x = 1,960$.

   ii. Viewing the Venn diagram above, each ‘area’ being non-negative requires $x \leq 2970$, $x \geq 500$ and $x \leq 4480$. Furthermore, as we have seen $x$ must be at least 1,960 or we have ‘too many customers’. Thus $1960 \leq x \leq 2970$.

![Figure 1.6: Venn diagram for Example 1.8.](image)
Example 1.9  An airline keeps information about its passengers and has noted the following facts about the services it supplied between London and New York during a particular week:

(a) The airline only operated two different types of aircraft which they denote by $A$ and $Z$.

(b) Travellers on $Z$ always have excess baggage, $X$, whereas travellers on $A$ sometimes do not.

(c) Smokers, $S$, always travel on $A$ and always have excess baggage.

(d) There is no Executive Class, $E$, travel on $Z$.

(e) Businessmen, $B$, always travel Executive Class and never smoke.

(f) Passengers requesting champagne, $C$, are always businessmen and never have excess baggage.

Interpret each of the above statements in set notation and hence construct a single Venn diagram to illustrate the relationships between $A$, $B$, $C$, $E$, $S$, $X$ and $Z$.

The following additional quantitative data are available for the week and route in question:

(g) 40% of all travellers on the airline used type $Z$ aircraft.

(h) Only 20% of all travelling businessmen did not request champagne.

(i) Businessmen make up 80% of all the executive class travellers.

(j) 150 smokers travelled with the airline. This represents 10% of all the travellers.

(k) There were 160 passengers who requested champagne.

What is the minimum and maximum number of non-smoking, non-executive class travellers on $A$ aircraft?
1.13 Summary

This chapter stands largely on its own as a topic. However, the concepts covered are extremely useful for summarising and depicting interrelated information. The topic is often thought of as being one of the easiest (and hence most popular from a candidate’s point of view). However, the translation from English to mathematical notation/diagrams (and vice versa) is a much underrated skill and needs thorough practice. The remarks included at the end of Section 1.15 might help you to establish the boundaries of the syllabus.

---

**Suggested solution:**

We have the following:

(a) \( A \cup Z = U \) (the universal set)

(b) \( Z \subseteq X \)

(c) \( S \subseteq (A \cap X) \)

(d) \( E \cap Z = \emptyset \) (the null or empty set)

(e) \( B \subseteq E \cap S^c \)

(f) \( C \subseteq B \cap X^c \)

[Note: there are alternative ways of depicting the above statements in set notation.]

The Venn diagram, taking into account all the above relationships, might look something like the one shown in Figure 1.8.

\[ n(S) = 150 = 10\% \] implies that \( n(\text{travellers}) = 1,500 \).
\[ n(C) = 160 \text{ and therefore } n(C)/n(B) = 0.8 \implies n(B) = 200. \]
Hence \( n(E) = 250 \) and \( n(Z) = 600, n(A) = 900. \)
If \( S \cap E = \emptyset \), then \( n(S^c \cap E^c \cap A) = 900 - (250 + 150) = 500 \) (minimum).
If \( n(S \cap E) = 50 \) (the most since \( B \cap S = \emptyset \)) then \( n(S^c \cap E^c \cap A) = 550 \) (maximum).

---

**Figure 1.8:** Venn diagram for Example 1.9.
1. Set theory

**What you do not need to know:**

- Anything about the strict definitions of open sets, closed sets.
- The extensions of set theory into group theory, rings, etc.
- The definition of sets of rational, irrational, real numbers, etc.
- The application of Venn diagrams and set theory in probability theory.

### 1.14 Solutions to activities

**Activity 1.1**

(a) \(\{a, b, c, d, e, f, g, h, j\}\)

(b) \(\{g\}\)

(c) \(\{b, c, d, f\}\)

(d) \(\{a, b, c, e, f\}\).

**Activity 1.2**

(a) \(A \cup (B \cap C^c)\) is shown in ‘i’.

(b) \((A \cup B \cup C)^c\) is shown in ‘ii’.

(c) \(B \cup A^c\) is shown in ‘iii’.

(d) \((A \cup B) \cap (B \cup C)\) is shown in ‘iv’.
Activity 1.3

(a) In total the number of businesses making claims = 20,000/100 = 200. However, the Venn diagram below is in ‘percentage of businesses making claims’. We let $x$ be the number of businesses making claims for all three perils.

Thus $40 + 45 + 13 + x = 100$, i.e. $98 + x = 100$ and therefore $x = 2\%$, i.e. 4 businesses.

(b) The total number of claims is

$$[27 + 32 + 15 + 2(3 + 13 + 8) + 3(2)] \times 2 = [74 + 2(24) + 6] \times 2 = 256.$$  

Activity 1.4

(a) If we let the number who are male with at least three years’ experience be $x$ then we can construct the Venn diagram below for $M$ (Males) and $E$ (at least three years’ experience) showing the order of the subsets (in terms of $x$).

Then, since every subset must have non-negative order, we must have

$$x \geq 6, \quad x \leq 16, \quad x \geq 0, \quad x \leq 10.$$  

Hence, in summary, $6 \leq x \leq 10$. The minimum number of males with at least three years’ experience is therefore six.

(b) Extending the above Venn diagram to include the set of graduates, $G$, we can argue as follows: the number of females is four, each of whom could have had at least three years’ experience and been graduates. Hence the maximum number of female graduates who have had at least three years’ experience is four. It can be checked by the following Venn diagram.
1. Set theory

1.15 A reminder of your learning outcomes

By the end of this chapter, and having completed the Essential reading and activities, you should be able to:

- understand the basic notation and nomenclature of sets especially: unions, intersections, complements, null sets, subsets, finite and infinite sets, differences of sets, the order of a set, the universal set

- construct a Venn diagram (not always straightforward!) from given relational data

- interpret Venn diagrams and set notation and explain their meaning in non-mathematical (‘everyday’) English

- use sets and Venn diagrams to analyse data (for example, to show inconsistencies and to derive maximum and/or minimum orders of sets).

1.16 Sample examination questions

1. (Please note that this is only part of a full examination question.)

The 120 employees of Union City Manufacturing are classified according to whether or not they are:

- skilled, $S$, or unskilled

- female, $F$, or male

- employed on the production line, $P$, or not.

<table>
<thead>
<tr>
<th>Group</th>
<th>Number of workforce</th>
<th>% of total salary bill</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>57</td>
<td>43</td>
</tr>
<tr>
<td>$P$</td>
<td>70</td>
<td>67</td>
</tr>
<tr>
<td>$S$</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>$F \cap P$</td>
<td>21</td>
<td>26</td>
</tr>
<tr>
<td>$F \cap S$</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$S \cap P$</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>$F \cap P \cap S$</td>
<td>$X$ (unknown)</td>
<td>$Y$ (unknown)</td>
</tr>
</tbody>
</table>
1.16. Sample examination questions

The table gives the number of employees which fall into each group identified, and also the percentage of the total salary bill paid to each group.

(a) From this table calculate the number of people (as a function of \(X\)) in each of the eight disjoint subsets which can be logically identified and produce an appropriate Venn diagram. Similarly produce a fully annotated Venn diagram for each group’s % of the total salary bill with subset orders as a function of \(Y\).

(b) Assuming that each subset of the above Venn diagrams has **positive and integer** order, determine the smallest possible value for \(X\) and the largest possible value for \(Y\).

(c) Assuming the values of \(X\) and \(Y\) determined in (b), which one of the eight subsets has the lowest salary per person?

2. A company undertakes a survey of its 120 adult employees and discovers that there are:

- 10 unmarried men without degrees
- 50 married employees
- 60 employees with degrees
- 30 unmarried women without degrees
- 20 women with degrees
- 15 married women.

(a) Draw a Venn diagram (with \(W\), \(D\), \(M\) denoting ‘women’, ‘has degree’ and ‘married’, respectively) in order to determine the maximum and minimum number of women who are married and have a degree.

(b) On the assumption that the number of married women with degrees takes its maximum value, construct a fully annotated Venn diagram (with \(W\), \(D\), \(M\) denoting ‘women’, ‘has degree’ and ‘married’, respectively) to show the order of each subset.

(c) Making use of the diagram in (b) above, describe each of the following subsets in words and state their order:

   i. \((W \cup M)^c\)
   ii. \((W^c \cap D^c \cap M)\)
   iii. \(M \cap (W \cup D^c)\).

3. An electronic subassembly consists of one of each of three components \(A\), \(B\) and \(C\) which are subject to faults. Tests have shown that failures of the subassemblies are
caused by faults in one, two or all three of the components \( A \), \( B \) and \( C \). Analysis of 10,000 subassemblies shows that 95% of the subassemblies are free from faults. Within the remainder there were 350 faulty \( A \) components, 250 faulty \( B \) components and 150 faulty \( C \) components. Of the subassemblies that failed, 220 were caused by failures in two components only, and (of these 220) 170 had faulty \( A \) components.

(a) Draw a Venn diagram to illustrate the above situation.  

(b) Create an equation for total component breakdowns and hence determine how many subassemblies tested had:
   i. faults in all three components at the same time.
   ii. no faulty \( B \) or \( C \) components?

(c) For each separate subset of your Venn diagram determine the maximum and minimum number of faulty assemblies within the subset consistent with all the information given in the question.

(d) If the subassembly repairs cost for \( A \), \( B \) and \( C \) components are respectively $5, $3 and $2, what are the maximum and minimum possible costs for repairing all the faulty components of the subassemblies tested which had faulty \( B \) components?

1.17 Guidance on answering the Sample examination questions

1. (a) We have the following Venn diagrams for workforce and % of total salary, respectively.
(b) Since each subset must have **positive and integer** order then $1 \leq X \leq 6$ and $1 \leq Y \leq 3$. [Note the difference between ‘positive’ and ‘nonnegative’.] Hence the minimum value of $X$ is 1, and the maximum value of $Y$ is 3.

(c) Using the above values for $X$ and $Y$ and evaluating the percentage of salaries per person for each of the eight subsets we can construct the following table:

<table>
<thead>
<tr>
<th>Subset</th>
<th>Workforce</th>
<th>% total salary</th>
<th>% salary ÷ workforce</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F \cap P^c \cap S^c$</td>
<td>30</td>
<td>12</td>
<td>0.40</td>
</tr>
<tr>
<td>$P \cap F^c \cap S^c$</td>
<td>42</td>
<td>40</td>
<td>0.95</td>
</tr>
<tr>
<td>$S \cap F^c \cap P$</td>
<td>4</td>
<td>11</td>
<td>2.75</td>
</tr>
<tr>
<td>$F \cap P \cap S^c$</td>
<td>20</td>
<td>23</td>
<td>1.15</td>
</tr>
<tr>
<td>$F \cap S \cap P^c$</td>
<td>6</td>
<td>5</td>
<td>0.83</td>
</tr>
<tr>
<td>$S \cap P \cap F^c$</td>
<td>7</td>
<td>1</td>
<td><strong>0.14</strong></td>
</tr>
<tr>
<td>$F \cap P \cap S$</td>
<td>1</td>
<td>3</td>
<td>3.00</td>
</tr>
<tr>
<td>$F^c \cap P \cap S^c$</td>
<td>10</td>
<td>5</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Hence the lowest salary per person (in bold) for subsets is $S \cap P \cap F^c$ (perhaps strangely).

2. (a) Drawing a Venn diagram of $W$, $D$ and $M$ and letting the number of married women with degrees be $x$ and the number of married men without degrees be $y$ gives the following Venn diagram.

Since the total number of employees is 120 then:

$$30 + (20 - x) + x + (15 - x) + y + (35 - y) + (5 + y) + 10 = 120,$$

i.e. $115 - x + y = 120$ and hence $y = 5 + x$. We can therefore rewrite the orders entirely in terms of $x$, as shown below.
1. Set theory

Noting that each subset must have non-negative order will produce the result that $0 \leq x \leq 15$ and hence the minimum order required is 0 and the maximum order is 15.

(b) & (c) Setting $x = 15$ gives the following Venn below and enables us to determine:

i. $n(W \cup M)^c = 35$

ii. $n(W^c \cap D^c \cap M) = 20$

iii. $n(M \cap (W \cup D^c)) = 35$.

This gives the following Venn diagram.

3. (a) Using the labels of areas (subsets) we obtain the following Venn diagram.

Areas $2 + 4 + 6 = 220$ subassemblies.

Areas $2 + 4 = 170$ subassemblies.

Hence area 6 = 50 subassemblies.
1.17. Guidance on answering the Sample examination questions

Letting \( n(A \cap B \cap C) = x \), we can then generate the Venn diagram for subassemblies as shown.

![Venn Diagram](image)

(b) Forming an equation for components broken we have:

\[
1 \times (\text{areas } 1 + 5 + 7) = 2 \times (\text{areas } 2 + 4 + 6) = 3 \times \text{area } 3 = 750,
\]

i.e. \( 280 - x + 2(220) + 3x = 750 \). Hence \( 720 + 2x = 750 \) and therefore \( x = 15 \).

i. 15 subassemblies had all three components faulty.

ii. \( 9500 + 180 - x = 9665 \) subassemblies have no faulty \( B \) or \( C \) components.

(c) \( n(A \cup B \cup C)^c = 9500 \) (given).

Area 1: \( n(A \cap B^c \cap C^c) = 165 \) (fixed).

Area 2: \( 85 \leq n(A \cap B \cap C^c) \leq 170 \).

Area 3: \( n(A \cap B \cap C) = 15 \) (fixed).

Area 4: \( 0 \leq n(A \cap B^c \cap C) \leq 85 \).

Area 5: \( 0 \leq n(A^c \cap B^c \cap C) \leq 85 \).

Area 6: \( n(A^c \cap B \cap C) = 50 \) (given).

Area 7: \( 15 \leq n(A^c \cap B \cap C^c) \leq 100 \).

The above may be obtained by looking at the extreme cases, shown below.

![Venn Diagram](image)
(d) Within $B$ areas 3 and 6 are ‘fixed’. Hence the above diagrams depict the ‘worst’ and ‘best’ cost situations for the problem posed, i.e.

\[
\begin{align*}
\text{max cost} & = (15)3 + (50)5 + (15)10 + (170)8 = $1,805 \\
\text{min cost} & = (100)3 + (50)5 + (15)10 + (85)8 = $1,380.
\end{align*}
\]
Chapter 2
Index numbers

2.1 Aims of the chapter

- To give a good (more than superficial!) understanding of the widely used techniques of index numbers.
- To establish what a wide range of alternative index construction methods are used and what a wide range of applications they have.
- To enable you to ask searching questions about the methodology used by producers, users and quoters of index numbers.

2.2 Learning outcomes

By the end of this chapter, and having completed the Essential reading and activities, you should be able to:

- understand how index numbers are created and for what reason
- work with all the following types of indices: price and quantity, simple, relative and aggregate, fixed base and chain-based, Paasche and Laspeyres, ideal and non-ideal
- create a deflated index
- link together indices with different bases
- fully interpret the message an index is telling you – this is an underrated skill
- choose an appropriate index to summarise a given set of data
- understand the advantages and disadvantages of the different index types
- appreciate the difficulties involved in choosing the best index for a given situation.

2.3 Essential reading

For this topic, the subject guide is sufficiently detailed and therefore there is no need to acquire books purely for the sake of index numbers.
2. Index numbers

2.4 Further reading

Of the texts listed in the introduction to this subject guide, both Jacques (Chapter 3) and Owen and Jones (Chapter 5) have reasonable coverage of index numbers. However, any other modern statistical text now tends to have a chapter on this increasingly used topic. Those particularly interested in the topic could refer to Allen, R.G.D. *Index numbers in theory and practice*. (Basingstoke: Palgrave Macmillan, 1982).

2.5 Introduction

In many ways this section of the subject stands apart from the rest. It is a self-contained topic with little or no overlap with other chapters in this subject introduction. However, index numbers are an increasingly used and much maligned phenomena of the present day world. All managers should appreciate the uses and abuses of Index numbers. It is included in the MT2076 Management mathematics syllabus both because of its growing importance for managers and because surprisingly few statistical courses devote sufficient time to the topic.

Indices are now used to measure a worker’s or company’s performance, the activities within financial markets, a country’s economic standing, etc. They are even used to determine the wage levels for certain types of workers.

Although the arithmetic of indices is simple, you will need to exercise great care in selecting the appropriate index to use and in performing the often tedious calculations involved. In addition, it is important for you to be able to interpret what (if anything!) a particular index value or series of index values is telling you.

2.6 The general approach and notation

‘An index’ is a statistical measure designed to show changes in a variable, or group of related variables, with respect to time. It shows the relative change rather than the absolute magnitude of change. There are many uses of index numbers (e.g. Financial Times Shares Index, Dow Jones Shares Index, Retail Price Index (RPI), index of industrial production, trade weighted depreciation figures, IQs, etc.). They have become increasingly popular over recent years however, they should be handled with care since it is easy to misconstrue their meaning and to make unrealistic assumptions when calculating them. Index numbers take various forms and vary from simple to complex. Some of the more common versions are given within this chapter.

It might be useful to indicate the general notation here:

\[ p_{it} = \text{price of ‘commodity’ } i \text{ in period } t \]
\[ q_{it} = \text{quantity of ‘commodity’ } i \text{ in period } t. \]

The term ‘commodity’ is used here to indicate the object(s) under consideration – it might be a car, a television, a basket of food, a week’s worth of labour, etc.
2.7 Simple index

This relates to one single commodity (and hence we may, if we wish, drop the suffix \(i\) here for simplicity).

\[
\text{Simple price index} = \frac{p_t \times 100}{p_0}
\]

where \(p_0\) is the price of the commodity in the ‘Base period’ (the period where the index is set to be 100 and the ‘yardstick’ against which other values are measured).

**Example 2.1** A simple index for labour costs per hour:

<table>
<thead>
<tr>
<th>Year</th>
<th>$ per hour ‘Price’ of labour</th>
<th>‘Price’ relative</th>
<th>Index (Base 2000 = 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>5.0</td>
<td>1.00</td>
<td>100.0</td>
</tr>
<tr>
<td>2001</td>
<td>5.2</td>
<td>1.04</td>
<td>104.0</td>
</tr>
<tr>
<td>2002</td>
<td>5.5</td>
<td>1.10</td>
<td>110.0</td>
</tr>
<tr>
<td>2003</td>
<td>6.0</td>
<td>1.20</td>
<td>120.0</td>
</tr>
<tr>
<td>2004</td>
<td>6.2</td>
<td>1.24</td>
<td>124.0</td>
</tr>
</tbody>
</table>

**Note:** The Base (2000 above) is often chosen to be as ‘normal’ as possible (i.e. when the price is not unduly high or low). The base period should be fairly up-to-date and consequently is updated periodically.

2.8 Simple aggregate index

This is used for a fixed group of ‘commodities’, say \(k\) in number. (The fixed quantities may be artificial i.e. not representing actual amounts purchased, etc.)

\[
\text{Simple aggregate price index for period } t = \frac{\sum_{i=1}^{k} p_{it}}{\sum_{i=1}^{k} p_{i0}} \times 100.
\]

**Example 2.2** A product cost index:

<table>
<thead>
<tr>
<th>‘Commodity’ input (i)</th>
<th>Quantity</th>
<th>2002 Base ‘prices’ (p_0)</th>
<th>2008 ‘prices’ (p_{it})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material A</td>
<td>1 kg</td>
<td>60</td>
<td>66</td>
</tr>
<tr>
<td>Material B</td>
<td>2 kg</td>
<td>30</td>
<td>48</td>
</tr>
<tr>
<td>Labour</td>
<td>3 hours</td>
<td>15</td>
<td>21</td>
</tr>
<tr>
<td>Overheads</td>
<td>3 hours</td>
<td>45</td>
<td>60</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td>150</td>
<td>195</td>
</tr>
</tbody>
</table>

Hence the simple aggregate price index for 2008 (Base 2002) is

\[
\frac{195}{150} \times 100 = 130.0.
\]
2. Index numbers

Note: One of the assumptions of the above index is that the quantities will remain the same throughout our analysis. This is obviously false in many situations. This difficulty can be tackled in various ways.

2.9 The average price relative index

In this index ‘commodities’ have equal importance. This has an advantage in that the index is independent of the quantities; however, the disadvantage is that it takes no account of the quantities!

Average price relative index for period \( t = \frac{1}{k} \sum_{i=1}^{k} \frac{p_{it}}{p_{i0}} \times 100. \)

Example 2.3 Continuing with Example 2.2, the price relatives for the four inputs are 1.1, 1.6, 1.4 and 1.33, respectively. The average price relative index is therefore

\[ (1.1 + 1.6 + 1.4 + 1.33) \times \frac{100}{4} = 110.8. \]

2.10 Weighted price relative indices

These use ‘weights’ \( w_i \) and form a weighted average of price relatives

\[
\frac{\sum_{i=1}^{k} w_i \cdot p_{it}/p_{i0}}{\sum_{i=1}^{k} w_i} \times 100.
\]

The weight \( w_i \) for commodity \( i \) is a measure of the importance of that commodity in the overall index. It might literally be the weight of commodity \( i \) used or purchased, or the number of units, total expenditure on that item in some period, etc.

2.10.1 Laspeyres’ (base period weighted) index

Here the relative weights for each item are calculated as the amount spent on each item in the base year, i.e. \( p_{i0}q_{i0} \).

Thus Laspeyres’ price index, named after Ernst Louis Étienne Laspeyres, for period \( t \) is

\[
\frac{\sum_{i=1}^{k} p_{i0}q_{i0} \cdot p_{it}/p_{i0}}{\sum_{i=1}^{k} p_{i0}q_{i0}} \times 100 = \frac{\sum_{i=1}^{k} p_{it}q_{i0}}{\sum_{i=1}^{k} p_{i0}q_{i0}} \times 100.
\]
2.10. Weighted price relative indices

Example 2.4  Continuing with Example 2.3, we might have:

<table>
<thead>
<tr>
<th>‘Commodity’ input $i$</th>
<th>‘Weight’ $w_{i} = p_{i0}q_{i0}$</th>
<th>Price relative $p_{it}/p_{i0}$</th>
<th>$w_{i} \times (p_{it}/p_{i0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material A</td>
<td>20</td>
<td>1.10</td>
<td>22.0</td>
</tr>
<tr>
<td>Material B</td>
<td>20</td>
<td>1.60</td>
<td>32.0</td>
</tr>
<tr>
<td>Labour</td>
<td>30</td>
<td>1.40</td>
<td>42.0</td>
</tr>
<tr>
<td>Overheads</td>
<td>50</td>
<td>1.33</td>
<td>66.6</td>
</tr>
<tr>
<td>Total</td>
<td>120</td>
<td></td>
<td>162.6</td>
</tr>
</tbody>
</table>

Hence Laspeyres’ price index for period

\[ t(2008) = \frac{162.2}{120} \times 100 = 135.2. \]

Note: If the quantities in the simple aggregate index correspond to the actual amount used for those inputs in the base period then the simple aggregate and Laspeyres’ indices give the same result.

2.10.2 Paasche’s (current period weighted) index

Here the relative weights for each item are calculated as the amount spent on each item in the current year at base period prices (i.e. $p_{i0}q_{it}$).

Thus Paasche’s price index, named after Hermann Paasche, for period $t$ is

\[
\frac{\sum_{i=1}^{k} p_{i0}q_{it} \cdot p_{it}/p_{i0}}{\sum_{i=1}^{k} p_{i0}q_{it}} \times 100 = \frac{\sum_{i=1}^{k} p_{it}}{\sum_{i=1}^{k} p_{i0}q_{it}} \times 100.
\]

Example 2.5  Obtaining and using the new weights $w_{i} = p_{i0}q_{it}$ for the product cost example:

<table>
<thead>
<tr>
<th>‘Commodity’ input $i$</th>
<th>‘Weight’ $w_{i} = p_{i0}q_{it}$</th>
<th>Price relative $p_{it}/p_{i0}$</th>
<th>$w_{i} \times (p_{it}/p_{i0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material A</td>
<td>30</td>
<td>1.10</td>
<td>33.0</td>
</tr>
<tr>
<td>Material B</td>
<td>40</td>
<td>1.60</td>
<td>64.0</td>
</tr>
<tr>
<td>Labour</td>
<td>40</td>
<td>1.40</td>
<td>56.0</td>
</tr>
<tr>
<td>Overheads</td>
<td>60</td>
<td>1.33</td>
<td>80.0</td>
</tr>
<tr>
<td>Total</td>
<td>170</td>
<td></td>
<td>233.0</td>
</tr>
</tbody>
</table>

Hence the Paasche’s index for the period $t$ is

\[ \frac{233}{170} \times 100 = 137.1. \]
2.10.3 Advantages and disadvantages of Laspeyres’ versus Paasche’s indices

The following table summarises the main advantages and disadvantages of these two important classes of indices.

<table>
<thead>
<tr>
<th>Laspeyres’ index</th>
<th>Paasche’s index</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Advantages</strong></td>
<td><strong>Advantages</strong></td>
</tr>
<tr>
<td>(a) Weights need calculating only once.</td>
<td>(a) Weights are up-to-date and more relevant</td>
</tr>
<tr>
<td>(b) As a consequence of (a) the calculation of the index is faster.</td>
<td></td>
</tr>
<tr>
<td><strong>Disadvantages</strong></td>
<td><strong>Disadvantages</strong></td>
</tr>
<tr>
<td>(a) Base weights quickly become irrelevant</td>
<td>(a) Collection of data to calculate latest weights may prove difficult.</td>
</tr>
<tr>
<td>(b) Index tends to overstate price increases since the weights are not altered to allow for movement from expensive items to cheaper ones.</td>
<td>(b) Price changes are under-estimated at the time of rising prices.</td>
</tr>
</tbody>
</table>

2.10.4 Other weights

For both Laspeyres’ and Paasche’s indices several ‘weights’ are possible, for example, use actual base or current period quantities (respectively) instead of outlays, i.e. use weights $q_{i0}$ and $q_{it}$ for the Laspeyres’ and Paasche’s price indices instead of $p_{i0}q_{i0}$ and $p_{i0}q_{it}$. These weights will produce relative rather than aggregate indices.

2.11 More complex, ‘ideal’ indices

The over- and under-estimation of price changes when using Laspeyres’ and Paasche’s indices has led to the concept of an ‘ideal’ index number, two of which are:

i. **Irving Fischer index:**

$$\sqrt{\frac{\sum_{i=1}^{k} p_{it}q_{i0}}{\sum_{i=1}^{k} p_{i0}q_{i0}}} \times \frac{\sum_{i=1}^{k} p_{it}q_{it}}{\sum_{i=1}^{k} p_{i0}q_{it}} \times 100$$

i.e. the geometric mean of the original indices.

ii. **Marshall-Edgeworth index:**

$$\frac{\sum_{i=1}^{k} p_{it}(q_{it} + q_{i0})/2}{\sum_{i=1}^{k} p_{i0}(q_{it} + q_{i0})/2} \times 100.$$
2.12 Volume indices

Strictly speaking, we have concentrated so far only on a price index. For each formulation of a price index the equivalent volume index may be expressed mathematically by replacing quantities with prices and vice versa. For example, the RPI is a price index whereas the Index of Industrial Production is a quantity or volume index.

We therefore have for example:

\[
\text{Laspeyres’ (aggregate) volume index} = \frac{\sum_{i=1}^{k} p_i q_{it}}{\sum_{i=1}^{k} p_i q_{i0}} \times 100.
\]

\[
\text{Paasche’s (aggregate) volume index} = \frac{\sum_{i=1}^{k} p_i q_{it}}{\sum_{i=1}^{k} p_i q_{i0}} \times 100.
\]

2.13 Index tests

The following two tests are often put forward as a means of determining how good an index is:

i. **Time reversal test**: the reversal of the time subscripts should produce the reciprocal of the original index, i.e. if the index calculates a value of 200 for the period \( t_2 \) when using a base of \( t_1 \), then it should ideally also give a value of 50 for the index in \( t_1 \) when using a base of \( t_2 \).

ii. **Factor reversal test**: the product of the price index and the quantity index should equal the index of total value, i.e.

\[
\frac{\sum_{i=1}^{k} p_i q_{it}}{\sum_{i=1}^{k} p_i q_{i0}} \times 100.
\]

Of those covered in this subject guide only the Irving Fischer index satisfies both the time reversal and factor reversal tests and is considered a truly ‘ideal index’.

2.14 Chain-linked index numbers

It is often argued that the base period must be regularly updated. As a consequence we have the chain-based index (very popular in the USA) which calculates the index required using the previous period as a base.
Thus we have Laspeyres’ (aggregate) chain price index

\[ \sum_{i=1}^{k} \frac{p_itq_{it-1}}{\sum_{i=1}^{k} p_{it-1}q_{it-1}} \times 100, \]

whereas Paasche’s (aggregate) chain price index is

\[ \sum_{i=1}^{k} \frac{p_itq_{it}}{\sum_{i=1}^{k} p_{it-1}q_{it}} \times 100. \]

Chain indices are particularly useful for period by period comparisons but, when considering a longer time period, indices with a single base are easier to interpret.

### 2.15 Changing a base and linking index series

Bases are often changed to make them more relevant. As a consequence you are often faced with the situation of having two or more indices apparently measuring the same thing but using different base periods. To produce a single index with a common base the most recent base period is usually chosen to be the base for the combined/linked series. Index values based on older bases are then adjusted for the change in base (see references for details). Although one is not mathematically justified in linking the series with different bases when the weights for individual commodities also change, the loss of precision is often small and the method acceptable.

### 2.16 ‘Deflating’ a series

With certain information, particularly when measured in monetary units, one might wish to adjust the data to produce ‘real’ figures which account for inflation, for example real earnings = apparent earnings / RPI.

**Activity 2.1** The table below shows two types of indices calculated over the period 2002 to 2007. The indices are obtained from the total value of output (given in billions of £) for a particular industrial sector in the UK, and the change in retail prices (RPI).

<table>
<thead>
<tr>
<th>Year</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index of output value (Base = 1997)</td>
<td>121.0</td>
<td>134.4</td>
<td>143.2</td>
<td>149.2</td>
<td>152.8</td>
<td>161.2</td>
</tr>
<tr>
<td>RPI</td>
<td>100</td>
<td>105</td>
<td>110</td>
<td>115</td>
<td>122</td>
<td>128</td>
</tr>
</tbody>
</table>

(a) Calculate a new index (with base year 2002) of the value of production output excluding the inflationary effects.
2.17 Further worked examples

Example 2.6 The following data represent the prices per unit of three different commodities in 2000 and 2005 and the total value of purchases in those years:

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Purchases (i.e. ( p_i q_i ))</th>
<th>Prices ( p_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>18</td>
<td>30</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>4.5</td>
</tr>
</tbody>
</table>

You are asked to construct price indices using (a) Paasche and (b) Laspeyres. [First note that here, and occasionally subsequently, we have dropped the commodity suffix ‘\( i \)’ and used a shorter notation for the summation over \( i \) too.] Since the question refers to expenditure on the three commodities, the weights are, in effect, value weights and hence must be multiplied by the price relatives, not just the prices in the two years. We therefore have:

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Prices</th>
<th>Value weights</th>
<th>Price relatives</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2000</td>
<td>2005</td>
<td>2000</td>
</tr>
<tr>
<td>A</td>
<td>6</td>
<td>5</td>
<td>18</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>32</td>
<td>54.5</td>
<td>34</td>
</tr>
</tbody>
</table>

(a) Paasche’s index is

\[
\frac{\sum_i w_i p_t / p_0}{\sum_i w_i} \times 100 = \frac{56.75}{54.5} \times 100 = 104.1.
\]

(b) Laspeyres’ index is

\[
\frac{\sum_i p_i n_i}{\sum_i p_{0i}} \times 100 = \frac{46 + 61 + 70 + 130}{\frac{45 + 60 + 80 + 120}{305}} \times 100 = \frac{30700}{305} = 100.656
\]

for April, and

\[
\frac{48 + 62 + 66 + 140}{305} \times 100 = \frac{31600}{305} = 103.607
\]

for May.
Example 2.7  A supplier of office furniture wishes to know if sales in real terms have increased in the 10-year period 1998–2008. Furthermore he would like to know if stock levels of his furniture were justified by the sales figures. The following data refer to the stock holdings of the suppliers four main furniture items at the end of 1998 and 2008:

<table>
<thead>
<tr>
<th>Items</th>
<th>1998 quantity $q_0$</th>
<th>Value $q_0p_0$ (£)</th>
<th>2008 quantity $q_t$</th>
<th>Value $q_tp_t$ (£)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chairs</td>
<td>400</td>
<td>40,000</td>
<td>300</td>
<td>60,000</td>
</tr>
<tr>
<td>Cabinets</td>
<td>700</td>
<td>80,000</td>
<td>900</td>
<td>180,000</td>
</tr>
<tr>
<td>Desks</td>
<td>140</td>
<td>42,000</td>
<td>200</td>
<td>90,000</td>
</tr>
<tr>
<td>Lights</td>
<td>60</td>
<td>30,000</td>
<td>90</td>
<td>60,000</td>
</tr>
</tbody>
</table>

Total sales for 1998 and 2008 were £1,200,000 and £2,400,000, respectively.

(a) Construct a weighted index of the price increases, 2008 as against 1998, for the four items of stock together.

(b) Calculate using the above index the percentage change of sales in real terms.

Suggested solution:

(a) First, determining the prices for 2008:

<table>
<thead>
<tr>
<th>Item</th>
<th>Number</th>
<th>£value</th>
<th>Price/unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chairs</td>
<td>300</td>
<td>60,000</td>
<td>200</td>
</tr>
<tr>
<td>Cabinets</td>
<td>900</td>
<td>180,000</td>
<td>200</td>
</tr>
<tr>
<td>Desks</td>
<td>200</td>
<td>90,000</td>
<td>450</td>
</tr>
<tr>
<td>Lights</td>
<td>90</td>
<td>60,000</td>
<td>666.67</td>
</tr>
</tbody>
</table>

And hence a Laspeyres’ price index can now be determined:

<table>
<thead>
<tr>
<th>Item</th>
<th>1998 quantity $q_0$</th>
<th>2008 prices $p_1$</th>
<th>$q_0p_1$</th>
<th>$q_0p_0$ (as given)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chairs</td>
<td>400</td>
<td>200</td>
<td>80,000</td>
<td>40,000</td>
</tr>
<tr>
<td>Cabinets</td>
<td>700</td>
<td>200</td>
<td>140,000</td>
<td>80,000</td>
</tr>
<tr>
<td>Desks</td>
<td>140</td>
<td>450</td>
<td>63,000</td>
<td>42,000</td>
</tr>
<tr>
<td>Lights</td>
<td>60</td>
<td>666.67</td>
<td>40,000</td>
<td>30,000</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>323,000</td>
<td>192,000</td>
</tr>
</tbody>
</table>

Hence index of price increases is

\[
(1998 = 100) = \frac{323000}{192000} \times 100 = 168.0
\]

(b) 1998 sales = £1,200,000 and hence using the index above the 1998 sales adjusted for price increases during the decade = £1,200,000 \times 168/100 = £2,016,000.

Hence real increase in sales = £(2,400,000 − 2,016,000) = £384,000, i.e. a 19.05% increase.
Example 2.8  Every month a company purchases four items in the typical quantities and at the prices shown below:

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Units</th>
<th>Weights</th>
<th>Price per units</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>March</td>
</tr>
<tr>
<td>W</td>
<td>Kilos</td>
<td>120</td>
<td>45</td>
</tr>
<tr>
<td>X</td>
<td>Kilos</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>Y</td>
<td>Litres</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>Z</td>
<td>Thousand</td>
<td>100</td>
<td>120</td>
</tr>
</tbody>
</table>

Using March as a base, find for April and May:

(a)  The simple aggregate price index.

(b)  The weighted aggregate price index.

(c)  If in June of the same year commodities W and X are expected to increase by one per cent per kilo and the price of commodity Z is expected to increase by 10 per cent per thousand, how much must the cost per litre of Y decrease in order that the weighted aggregate price index for June remains the same as for May?

Suggested solution:

(a)  Simple aggregate index (March = 100)

\[
\frac{\sum w_i (p_i/p_0)}{\sum w_i} \times 100 = \frac{34}{32} \times 100 = 106.2.
\]

(b)  A weighted aggregate price index is obtained by using the weights 120, 50, 60 and 100. Although it should be borne in mind that a number of other possible weighted indices are possible, the following seems reasonable. For April,

\[
\frac{\sum w_i (p_i/p_0)}{\sum w_i} \times 100 = \frac{120(46/45) + 50(61/60) + 60(70/80) + 100(130/120)}{330} \times 100
\]

\[
= \frac{334.333}{330} \times 100 = 101.31.
\]

Similarly a weighted aggregate price index for May is \(= 100 \times \frac{345.83}{330} = 104.79\).

(c)  If the index is to remain as before, then

\[
\frac{\sum w_i \cdot p_i}{p_0}
\]

must remain unchanged, i.e.

\[
345.83 = 120(48.48/45) + 50(62.62/60) + 60(1 - y)(66/80) + 100(154/120)
\]
(here 100y is the percentage decrease in the price of Y).

Solving for $y$:

$$49.5(1 - y) = 345.83 - 129.28 - 52.18 - 128.33 = 36.04,$$

i.e. $y = 0.272$. So there is a price decrease of 27.20% in the May price for Y.

[Note: A completely different set of results for (b) and (c) is possible if an aggregate index of

$$\frac{\sum_i w_i p_{i1}}{\sum_i w_i p_{i0}} \times 100$$

is formed. In this case the answers become (b) 102.3 and 106.4 and (c) 19.5%. This demonstrates the ability to create and use many apparently acceptable indices.]

### 2.18 The practical problems of selecting an appropriate index

In the above notes we have seen some of the very wide range of index numbers that might be calculated. This is an indication of the complexity one is faced with when trying to decide upon an appropriate index structure. There are certain key facts which one must establish before the appropriate index can be chosen. They include:

- Why do we require an index? Is it to represent changing prices, changing quantities or changing expenditures?
- Do we prefer a fixed base or an updated base methodology?
- What are the costs and time delays of acquiring the data?
- How many ‘commodities’ should we include? Which representative ‘commodities’ should we use?
- How often should we collect the data?
- What weights should we use? How often should they be updated?
- Should we deflate the index?

On the next few pages are tables which indicate the sort of complexity involved once one has decided to construct an index, in this case an index for share prices (e.g. on the London Stock Exchange). Table 2.1 summarises three of the main indices that have been used over many years to measure share prices in the stock exchange, namely the FTSE 100 Index, The FT ‘All Share’ Index and the FT 250 Index. Even if one decides to use 100 chosen shares, which ones do we choose? The list in 1989 (shown in Table 2.2) was a representative mixture of companies from various industrial/economic sectors. For comparison, the current (January 2008) list of FTSE 100 companies is shown in Table 2.3 and you should note how different the list has become. This is not
2.18. The practical problems of selecting an appropriate index

<table>
<thead>
<tr>
<th>Base date</th>
<th>FTSE 100</th>
<th>FT All Share Index</th>
<th>FT 250 Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index construction</td>
<td>Market free-float capitalisation weighted arithmetic average</td>
<td>Market free-float capitalisation weighted arithmetic average</td>
<td>Market free-float capitalisation weighted arithmetic average</td>
</tr>
<tr>
<td>Number of constituents</td>
<td>100</td>
<td>683</td>
<td>250</td>
</tr>
<tr>
<td>Market coverage</td>
<td>80–85% of the entire UK equity market</td>
<td>Approximately 98% of the entire UK equity market</td>
<td>Approximately 14–15% of the entire UK equity market</td>
</tr>
<tr>
<td>Calculation frequency</td>
<td>Effectively continuously (every 15 seconds 09:00–17:00 daily) and end of day</td>
<td>Effectively continuously (every 60 seconds 09:00–17:00 daily) and end of day</td>
<td>Whenever a change in the price of one of the stock occurs.</td>
</tr>
<tr>
<td>Review of constituents</td>
<td>Every three months in March, June, September, December.</td>
<td>See if you can find out – merely for interest.</td>
<td>See if you can find out – merely for interest.</td>
</tr>
</tbody>
</table>

Table 2.1: A comparison of the characteristics of some London share indices, as at January 2008.

Surprising as the index membership is reviewed every three months. If interested, see www.ftse.com/Indices/index.jsp for more details. The ‘weights’ that are used are constantly changing to give more or less importance to certain shares. These tables constitute one index measure at moments in time. Many of the shares chosen to be within the index at one time have been replaced with new ones; the weights (capitalisations) have changed also. Certain companies are no longer so important, some have been taken over, some new companies have been created by privatisation, etc. Some companies have moved in or out of the FTSE 100 on several occasions. The problems are immense.
## Table 2.2: The industrial/economic sector breakdown of the 100 share in the FTSE 100 in 1989.

<table>
<thead>
<tr>
<th>Builders/Construction</th>
<th>Food Retailing</th>
<th>Oil &amp; Gas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue Circle Industries</td>
<td>Argyll Group</td>
<td>British Petroleum</td>
</tr>
<tr>
<td>BPB Industries</td>
<td>ASDA MFI</td>
<td>British Gas</td>
</tr>
<tr>
<td>English China Clays</td>
<td>Sainsbury</td>
<td>Burmah Oil</td>
</tr>
<tr>
<td>Pilkington</td>
<td>Tesco</td>
<td>Enterprise Oil</td>
</tr>
<tr>
<td>Redland</td>
<td></td>
<td>LASMO</td>
</tr>
<tr>
<td>RMC</td>
<td></td>
<td>Shell</td>
</tr>
<tr>
<td>Tarmac</td>
<td></td>
<td>Ultrasan</td>
</tr>
<tr>
<td>Taylor Woodrow</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Electrical/Electronics</strong></td>
<td><strong>Health &amp; Household</strong></td>
<td><strong>Banks &amp; Other</strong></td>
</tr>
<tr>
<td>BICC</td>
<td>British Oxygen</td>
<td><strong>Financial</strong></td>
</tr>
<tr>
<td>Carlton Communications</td>
<td>Fisons</td>
<td>Abbey National</td>
</tr>
<tr>
<td>GEC</td>
<td>Glaxo</td>
<td>Barclays Bank</td>
</tr>
<tr>
<td>Lucas</td>
<td>ICI</td>
<td>Lloyds Bank</td>
</tr>
<tr>
<td>Racal</td>
<td>Reckitt &amp; Colman</td>
<td>Midland Bank</td>
</tr>
<tr>
<td><strong>Brewers, Distillers and Leisure</strong></td>
<td>Smith &amp; Nephew</td>
<td>NatWest Bank</td>
</tr>
<tr>
<td>Allied Lyons</td>
<td>SmithKline Beecham</td>
<td>Royal Bank of Scotland</td>
</tr>
<tr>
<td>Bass</td>
<td>Wellcome</td>
<td>Standard Chartered</td>
</tr>
<tr>
<td>Grand Metropolitan</td>
<td></td>
<td>TSB</td>
</tr>
<tr>
<td>Guinness</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ladbroke</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scottish &amp; Newcastle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trushouse Forte</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Whitbread A</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Misc Holding Companies</strong></td>
<td><strong>Stores</strong></td>
<td><strong>Insurance Companies</strong></td>
</tr>
<tr>
<td>BAA</td>
<td>Boots</td>
<td>Commercial Union</td>
</tr>
<tr>
<td>BAT Industries</td>
<td>Burton</td>
<td>General Accident</td>
</tr>
<tr>
<td>British Airways</td>
<td>GUS A</td>
<td>Guardian Royal Exchange</td>
</tr>
<tr>
<td>BTR</td>
<td>Kingfisher</td>
<td>Legal &amp; General</td>
</tr>
<tr>
<td>Cookson</td>
<td>Marks &amp; Spencer</td>
<td>Prudential</td>
</tr>
<tr>
<td>Granada</td>
<td>Sears</td>
<td>Royal Insurance</td>
</tr>
<tr>
<td>Lonrho</td>
<td></td>
<td>Sun Alliance</td>
</tr>
<tr>
<td>Hanson Trust</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pearson</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P&amp;O Deferred</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Polly Peck</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rank Organisation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reuters B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rothmans</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thorn EMI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trafalgar House</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Food Manufacturing</strong></td>
<td><strong>Telecommunications</strong></td>
<td><strong>Mining Finance</strong></td>
</tr>
<tr>
<td>Assoc. British Foods</td>
<td>British Telecom</td>
<td>Rio Tinto Zinc</td>
</tr>
<tr>
<td>Cadbury</td>
<td>Cable &amp; Wireless</td>
<td></td>
</tr>
<tr>
<td>Hillsdown</td>
<td>STC</td>
<td></td>
</tr>
<tr>
<td>RHM Group</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unilever</td>
<td></td>
<td></td>
</tr>
<tr>
<td>United Biscuits</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Property</strong></td>
<td><strong>Textiles</strong></td>
<td><strong>Engineering</strong></td>
</tr>
<tr>
<td>Hammerson A</td>
<td>Courtaulds</td>
<td><strong>BET</strong></td>
</tr>
<tr>
<td>Land Securities</td>
<td></td>
<td>British Steel</td>
</tr>
<tr>
<td>MEPC</td>
<td></td>
<td>British Aerospace</td>
</tr>
<tr>
<td><strong>Engineering</strong></td>
<td></td>
<td>GKN</td>
</tr>
<tr>
<td><strong>Paper &amp; Packaging</strong></td>
<td></td>
<td>Hawker Siddeley</td>
</tr>
<tr>
<td>Maxwell Communications</td>
<td></td>
<td>Rolls Royce</td>
</tr>
<tr>
<td>Reed International</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Textiles</strong></td>
<td></td>
<td><strong>Siebe</strong></td>
</tr>
<tr>
<td><strong>Mining Finance</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rio Tinto Zinc</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Nestle</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Engineering</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Food Manufacturing</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Activity 2.2  (Mainly for interest) Compare Table 2.2 with Table 2.3 and note how few companies were in the FTSE 100 in 1989 and 2008. Note also the changing type of company involved.

Activity 2.3  (Mainly for interest) Try to find out what the weightings are for the 100 companies in the FTSE 100 – the constituents might have changed by the time you read this subject guide (remember the company list is reviewed every three months and often several changes occur).

2.19  Summary

What you should know

The subject of index numbers is wide-ranging due to the many alternative indices which can be created from a data stream – you may come across some extra ones that are not specifically mentioned within this subject guide. However, this chapter refers to all the index types you are called upon to understand and use within this course.

What you do not need to know

There is no obligation for you to know how any particular ‘well-known’ index (for example, the Financial Times 100 or the Dow Jones) has been created. However, it is important to understand the difficulties in constructing indices that have such aims.

2.20  Solution to Activity 2.1

To produce an index with 2002 as a base we calculate \( \left( \frac{I_t}{I_{2002}} \right) \times 100 \). We then deflate this series by dividing the series just obtained by the (RPI/100) for the corresponding year. Since the RPI has a value of 100.0 in 2002 there will be no further adjustments required:

(a)  We have:

<table>
<thead>
<tr>
<th>Year</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index of Output Value ( I_t ) (Base = 1997)</td>
<td>121.0</td>
<td>134.4</td>
<td>143.2</td>
<td>149.2</td>
<td>152.8</td>
<td>161.2</td>
</tr>
<tr>
<td>RPI</td>
<td>100.0</td>
<td>105.0</td>
<td>110.0</td>
<td>115.0</td>
<td>122.0</td>
<td>128.0</td>
</tr>
<tr>
<td>Index of Output Value (Base = 2002)</td>
<td>100.0</td>
<td>111.07</td>
<td>118.35</td>
<td>123.31</td>
<td>126.28</td>
<td>133.22</td>
</tr>
<tr>
<td>Deflated Index series for Output Value (Base = 2002)</td>
<td>100.00</td>
<td>105.79</td>
<td>107.59</td>
<td>107.22</td>
<td>103.51</td>
<td>104.08</td>
</tr>
</tbody>
</table>

(b)  The greatest annual percentage increase is 5.79% between 2002 and 2003.
Table 2.3: The January 2008 list of companies in the FTSE 100.
2.21 A reminder of your learning outcomes

By the end of this chapter, and having completed the Essential reading and activities, you should be able to:

- understand how index numbers are created and for what reason
- work with all the following types of indices: price and quantity, simple, relative and aggregate, fixed base and chain-based, Paasche and Laspeyres, ideal and non-ideal
- create a deflated index
- link together indices with different bases
- fully interpret the message an index is telling you – this is an underrated skill
- choose an appropriate index to summarise a given set of data
- understand the advantages and disadvantages of the different index types
- appreciate the difficulties involved in choosing the best index for a given situation.

2.22 Sample examination questions

1. The costs per kilogram of raw material $X$ and $Y$ have been registered from 2000 to 2006 and are reproduced below:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cost per kg $X$</th>
<th>Cost per kg $Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>7.00</td>
<td>4.00</td>
</tr>
<tr>
<td>2001</td>
<td>7.35</td>
<td>4.20</td>
</tr>
<tr>
<td>2002</td>
<td>7.98</td>
<td>4.70</td>
</tr>
<tr>
<td>2003</td>
<td>8.61</td>
<td>4.10</td>
</tr>
<tr>
<td>2004</td>
<td>9.10</td>
<td>5.10</td>
</tr>
<tr>
<td>2005</td>
<td>9.73</td>
<td>5.40</td>
</tr>
<tr>
<td>2006</td>
<td>10.43</td>
<td>5.60</td>
</tr>
</tbody>
</table>

The Multimix company has used $X$ and $Y$ in its product XANDY in the proportions 40:60 by weight throughout the above period.

(a) Produce separate material price indices (Base 2000 = 100) for the raw materials $X$ and $Y$.

(b) Construct a chain-based unlinked index for the raw material $X$ and illustrate its usefulness by determining the year in which the greatest percentage increase in the price of $X$ occurred. What is the size of this increase?

(4 marks)
2. Index numbers

(c) Construct an index series (Base 2000 = 100) for the total material cost of XANDY. Comment upon this series.

(6 marks)

(d) Assuming that the costs of X and Y will continue to increase in the future at a rate equal to their average rates of increase over the period 2000 to 2006, what prediction would you give for the XANDY total material cost index in 2008?

(6 marks)

2. (Please note that this question is only part of a full examination question.)

The following table gives indices for share prices on a stock exchange using two different index methods (the collected share index based in 1985 and the illustrative share index based in 2005). Also given is an inflation index (Base 1975).

<table>
<thead>
<tr>
<th>Year</th>
<th>‘Collected’ Index (Base 1985 = 100)</th>
<th>‘Illustrative’ Index (Base 2005 = 100)</th>
<th>Inflation Index (Base 1975 = 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>145.0</td>
<td></td>
<td>310.0</td>
</tr>
<tr>
<td>2003</td>
<td>158.1</td>
<td></td>
<td>315.6</td>
</tr>
<tr>
<td>2004</td>
<td>170.2</td>
<td></td>
<td>330.4</td>
</tr>
<tr>
<td>2005</td>
<td>188.2</td>
<td>100.0</td>
<td>358.0</td>
</tr>
<tr>
<td>2006</td>
<td></td>
<td>116.9</td>
<td>378.8</td>
</tr>
<tr>
<td>2007</td>
<td></td>
<td>146.0</td>
<td>383.4</td>
</tr>
<tr>
<td>2008</td>
<td></td>
<td>168.0</td>
<td>410.4</td>
</tr>
</tbody>
</table>

Using the above data you are asked to:

(a) Combine the two index series for share prices so that the resultant series has a common base.

(4 marks)

(b) Produce a series of deflated share prices to indicate whether share prices have gone up more or less than inflation. In which year was the highest percentage increase in deflated share prices and what was the value of this increase?

(8 marks)

3. (a) An economic leading indicator is designed to move up or down before the economy begins to move the same way. Suppose you want to construct a leading economic indicator. Because of the time and work involved, you decide to use only four time series. You select the following four series: unemployment, stock prices, producer prices and exports. Here are the figures for 1989 and 1991:

<table>
<thead>
<tr>
<th>Economic Time Series</th>
<th>1989</th>
<th>1991</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment rate (%)</td>
<td>5.3</td>
<td>6.8</td>
</tr>
<tr>
<td>Index of stock prices, Standard &amp; Poors (1942 = 100)</td>
<td>265.88</td>
<td>362.26</td>
</tr>
<tr>
<td>Producer Price Index (1984 = 100)</td>
<td>109.6</td>
<td>115.2</td>
</tr>
<tr>
<td>Exports ($1,000 millions)</td>
<td>529.9</td>
<td>622.8</td>
</tr>
</tbody>
</table>

The weights you arbitrarily assign are: Unemployment Rate 20%, Stock Prices 40%, Producers Price Index 25% and Exports 15%.

Using 1989 as the base period, construct a leading economic indicator (index value) for 1991. Interpret your leading indicator.

(6 marks)

(b) Prices of selected foods for 1977 and 1992 are given in the following table:

<table>
<thead>
<tr>
<th>Item</th>
<th>1977</th>
<th>1992</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
<td>Amount Produced</td>
</tr>
<tr>
<td>Cabbage</td>
<td>6</td>
<td>2000</td>
</tr>
<tr>
<td>Carrots</td>
<td>10</td>
<td>200</td>
</tr>
<tr>
<td>Peas</td>
<td>20</td>
<td>400</td>
</tr>
<tr>
<td>Lettuce</td>
<td>15</td>
<td>100</td>
</tr>
</tbody>
</table>

i. Using the Laspeyres’ formula, calculate a weighted index of price for 1992 (1977 = 100).

(4 marks)


(4 marks)

iii. Interpret each of the two price indices above and discuss the appropriateness of each.

(2 marks)


(4 marks)

2.23 Guidance on answering the Sample examination questions

1. (a) & (b)

<table>
<thead>
<tr>
<th>Year</th>
<th>Cost per kg X</th>
<th>Cost per kg Y</th>
<th>Index for X</th>
<th>Index for Y</th>
<th>Chained index for X</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>7.00</td>
<td>4.00</td>
<td>100.0</td>
<td>100.0</td>
<td>105.0</td>
</tr>
<tr>
<td>2001</td>
<td>7.35</td>
<td>4.20</td>
<td>105.0</td>
<td>105.0</td>
<td>105.0</td>
</tr>
<tr>
<td>2002</td>
<td>7.98</td>
<td>4.70</td>
<td>114.0</td>
<td>117.5</td>
<td>108.57</td>
</tr>
<tr>
<td>2003</td>
<td>8.61</td>
<td>4.10</td>
<td>123.0</td>
<td>102.5</td>
<td>107.89</td>
</tr>
<tr>
<td>2004</td>
<td>9.10</td>
<td>5.10</td>
<td>130.0</td>
<td>127.5</td>
<td>105.69</td>
</tr>
<tr>
<td>2005</td>
<td>9.73</td>
<td>5.40</td>
<td>139.0</td>
<td>135.0</td>
<td>106.92</td>
</tr>
<tr>
<td>2006</td>
<td>10.43</td>
<td>5.60</td>
<td>149.0</td>
<td>140.0</td>
<td>107.19</td>
</tr>
</tbody>
</table>

Greatest increase in X per year is for 2002 when the rise was 8.57%.
2. Index numbers

(c)

<table>
<thead>
<tr>
<th>Year</th>
<th>XANDY per kg</th>
<th>XANDY Price Index</th>
<th>XANDY chained</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>5.20</td>
<td>100.0</td>
<td></td>
</tr>
<tr>
<td>2001</td>
<td>5.46</td>
<td>105.0</td>
<td>105.0</td>
</tr>
<tr>
<td>2002</td>
<td>6.01</td>
<td>115.6</td>
<td>110.1</td>
</tr>
<tr>
<td>2003</td>
<td>5.90</td>
<td>113.5</td>
<td>98.2</td>
</tr>
<tr>
<td>2004</td>
<td>6.70</td>
<td>128.8</td>
<td>113.5</td>
</tr>
<tr>
<td>2005</td>
<td>7.13</td>
<td>137.2</td>
<td>106.4</td>
</tr>
<tr>
<td>2006</td>
<td>7.53</td>
<td>144.8</td>
<td>105.6</td>
</tr>
</tbody>
</table>

The price of XANDY is increasing other than in year 2003. The largest increase is in 2004 when it increased by 13.5%.

(d) Average annual percentage increase in X is \((149 - 100)/6 = 8.1667\%\) and average annual percentage increase in Y is 6.6667%. Hence we would predict that in 2008, X will cost \((10.43 \times (1.081667)^2 = 12.20\) per kg and Y will cost \((5.6) \times (1.066667)^2 = 6.37\) per kg. Hence XANDY is predicted to cost \((12.20 \times 0.4) + (6.37 \times 0.6) = 8.70\) per kg.

[Alternatively (and probably more reasonably), using compound increases, annual percentage increase in X, say \(x\), is such that \((1 + x)^6 = 149/100\) i.e. \(x = 1.49^{1/6} - 1 = 0.0687\). Similarly we might determine the annual (compound) increase in Y, say \(y\), as \(y = 1.40^{1/6} - 1 = 0.0577\).]

Hence in 2008, X will cost \((10.43 \times (1.0687)^2 = 11.91\) per kg and Y will cost \((5.6)(1.0577)^2 = 6.26\) per kg. Hence XANDY is predicted to cost \((11.91 \times 0.4) + (6.26 \times 0.6) = 8.52\) per kg.]

2.

(a) & (b) To combine the two series we multiply the base 1985 values by the conversion factor 100/188.2 (i.e. the one year of overlap gives a measure of the relative values of the two indices). By convention we would pick the later of the two bases for the combined index. Afterwards we would deflate the series by multiplying by 100/(inflation index) for each year. Then (perhaps) form a new index series for deflated share prices with 2002 = 100 (by multiplying by 100/24.855). We therefore get the following results:

<table>
<thead>
<tr>
<th>Year</th>
<th>Combined Index</th>
<th>Inflation Index</th>
<th>Deflated Share Price</th>
<th>Deflated Index (Base 2002 = 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>77.05</td>
<td>310.0</td>
<td>24.855</td>
<td>100.0</td>
</tr>
<tr>
<td>2003</td>
<td>84.01</td>
<td>315.6</td>
<td>26.619</td>
<td>107.10</td>
</tr>
<tr>
<td>2004</td>
<td>90.44</td>
<td>330.4</td>
<td>27.373</td>
<td>110.13</td>
</tr>
<tr>
<td>2005</td>
<td>100.0</td>
<td>358.0</td>
<td>27.933</td>
<td>112.38</td>
</tr>
<tr>
<td>2006</td>
<td>116.9</td>
<td>378.8</td>
<td>30.861</td>
<td>124.16</td>
</tr>
<tr>
<td>2007</td>
<td>146.0</td>
<td>383.4</td>
<td>38.080</td>
<td>153.21</td>
</tr>
<tr>
<td>2008</td>
<td>168.0</td>
<td>410.4</td>
<td>40.936</td>
<td>164.70</td>
</tr>
</tbody>
</table>

The highest percentage increase in deflated share prices occurred in 2007 when it rose by \((153.21 - 124.16)/124.16 = 23.4\%\). A chain index might show this even more clearly.

50
3. (a) Indicator =

\[ \sum w_i \left( \frac{p_{it}}{p_i} \right) / \sum w_i = 0.2 \left( \frac{6.8}{5.3} \right) + 0.4 \left( \frac{362.26}{265.88} \right) + 0.25 \left( \frac{115.2}{109.6} \right) + 0.15 \left( \frac{622.8}{529.9} \right) = 1.2404. \]

i.e. an index of 124.04.

So the leading economic indicator has increased in value from 1 in 1989 to 1.2404 in 1991. Business activity increased 24% from 1989 to 1991.

Least impact is caused by Exports which rose by only 17.5% with weight of 15%.

(b) i. Laspeyres’ price index =

\[ \frac{\sum p_{it}q_{i0}}{\sum p_{i0}q_{i0}} \times 100 = \frac{(5)(2000) + (12)(200) + (18)(400) + (15)(100)}{(6)(2000) + (10)(200) + (20)(400) + (15)(100)} \times 100 = 89.8 \]

i.e. prices down by 10.2%.

ii. Paasche’s price index =

\[ \frac{\sum p_{it}q_{it}}{\sum p_{i0}q_{it}} \times 100 = \frac{(5)(1500) + (12)(200) + (18)(500) + (15)(200)}{(6)(1500) + (10)(200) + (20)(500) + (15)(200)} \times 100 = 91.3 \]

i.e. price down by 8.7%.

iii. Appropriateness depends upon cost of acquiring latest quantity data, degree of changing tastes/substitution, desire for accuracy etc.

\[ \text{Value index} = \frac{\sum p_{it}q_{it}}{\sum p_{i0}q_{i0}} \times 100 = 93.2. \]

2. Index numbers
Chapter 3
Trigonometric functions and imaginary numbers

3.1 Aims of the chapter

- To indicate how trigonometric functions can be used to model dynamic (or static) situations where cycles are present.
- To explain how imaginary numbers can occur as the solution to certain quadratic equations.
- To establish a relationship between complex numbers, exponential and trigonometric functions.
- To provide a solid mathematical basis for some of the problems encountered when solving difference or differential equations.

3.2 Learning outcomes

By the end of this chapter, and having completed the Essential reading and activities, you should be able to:

- evaluate the values of trigonometric functions
- sketch graphs of the three main trigonometric functions and functions of them
- differentiate and integrate functions involving trigonometric functions
- use series expansions of trigonometric functions and exponentials
- manipulate and use imaginary numbers
- use De Moivre’s theorem to interchange between complex numbers and trigonometric functions
- represent complex numbers on an Argand diagram
- understand the meaning and usefulness of complex conjugates.
3.3 Essential reading


3.4 Further reading


3.5 Introduction

Sines, cosines and tangents are functions which one learns at school, where they are mainly taught as a means of solving geometric problems concerning triangles. Although this is clearly an important application of such trigonometric functions, more important for a manager and mathematical modeller is the use of such functions in dynamic relationships (e.g. in describing economic cycles, competitive markets, etc.). These applications occur because of the cyclical nature of these trigonometric functions. They are particularly useful in solving certain difference and differential equations but before embarking upon these important areas (Chapters 4 and 5) we must first learn (or perhaps simply recall) the basics of trigonometric functions.

Related to this area is the topic of imaginary numbers. It seems strange that a whole new number system involving the concept of an imaginary number \( i = \sqrt{-1} \) is very important for modelling and system investigations for economists and management sciences. However, imaginary numbers are extremely useful in the field of mathematics and, although it is not the intention of this course to turn you into mathematicians, they are sufficiently important that their basic ideas and usefulness should be part of this second/supplementary mathematics course.

Perhaps this chapter is more theoretical in nature than we would initially wish. However, by means of suitable economic and management models we hope to demonstrate their usefulness in due course. Furthermore, as already stated, this chapter is a necessary prerequisite for certain aspects of Chapters 4 and 5 on difference and differential equations.

3.6 Basic trigonometric definitions and graphs (a reminder)

Consider the following right-angled triangle \( ABC \) in Figure 3.1.

The **sine** (abbreviated sin) of the angle \( \theta \) i.e. \( \sin \theta \) is defined as \( a/c \).

The **cosine** (abbreviated cos) of the angle \( \theta \) i.e. \( \cos \theta \) is defined as \( b/c \) and the **tangent** (abbreviated tan) of the angle \( \theta \) i.e. \( \tan \theta \) is defined as \( a/b \).
3.6. Basic trigonometric definitions and graphs (a reminder)

For any angle $\theta$, $\sin \theta$ is finite and takes values between $-1$ and $+1$ (inclusive of these limiting values). A similar statement holds for $\cos \theta$. For $\tan \theta$, however, we can have values anywhere between $-\infty$ and $+\infty$.

The graphs of these trigonometric functions are given in Figures 3.2, 3.3 and 3.4, respectively.

The angles can be defined in terms of degrees ($^\circ$) or radians (Figures 3.2 to 3.4) above use $\pi$ radians for the horizontal axes. A radian is defined as the angle subtended by an arc of length 1 in a circle of radius 1. Thus an angle of $x$ radians in a circle of radius $r$ is subtended by an arc of length $rx$ (see Figure 3.5).

Recognising that the circumference of the circle is $2\pi r$, where $\pi = 3.1416$ approximately, then $60^\circ = \pi/3$ radians, $90^\circ = \pi/2$ radians, $180^\circ = \pi$ radians and $360^\circ = 2\pi$ radians etc. Although it is possible to work with either degrees or radians within this and many other courses involving trigonometric functions, many of the
3. Trigonometric functions and imaginary numbers

Figure 3.3: Plot of the cos function.

Figure 3.4: Plot of the tan function.
application area and texts tend to use radians. This is a practice which this subject
guide will normally follow (although you are perfectly free to use degrees if you prefer).

Values for sin, cos and tan of an angle will be found on all but the most basic
calculators. However, partly (but not entirely) as only a basic calculator is allowed
in the examination, certain values are worth remembering. For example:

<table>
<thead>
<tr>
<th>Angle (θ radians)</th>
<th>Angle (°)</th>
<th>Sine θ</th>
<th>Cosine θ</th>
<th>Tangent θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>π/6</td>
<td>30</td>
<td>1/2</td>
<td>√3/2</td>
<td>1/√3</td>
</tr>
<tr>
<td>π/4</td>
<td>45</td>
<td>1/√2</td>
<td>1/√2</td>
<td>1</td>
</tr>
<tr>
<td>π/3</td>
<td>60</td>
<td>√3/2</td>
<td>1/2</td>
<td>√3</td>
</tr>
<tr>
<td>π/2</td>
<td>90</td>
<td>1</td>
<td>0</td>
<td>∞</td>
</tr>
<tr>
<td>π</td>
<td>180</td>
<td>0</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>3π/2</td>
<td>270</td>
<td>−1</td>
<td>0</td>
<td>−∞</td>
</tr>
<tr>
<td>2π</td>
<td>360</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Activity 3.1**  This activity should essentially be revisiting material you have
already covered in your 100 courses. State the values of each of the following
trigonometric functions without the use of a calculator (you may use surds, i.e.
square roots, where necessary):

(a)  \(\cos(5\pi/3), \sin(-\pi/6), \tan(7\pi/3), \sin(11\pi/3), \cos(3\pi/4), \sin(7\pi/6), \tan(7\pi/4)\)
where the angles are in radians.

(b)  \(\cos(135), \tan(-45), \sin(225), \cos(-45), \sin(300), \tan(420)\) where the angles are in degrees.
3. Trigonometric functions and imaginary numbers

Activity 3.2  Produce a sketch diagram for each of the following trigonometric functions:

(a) 3 cos(\(\pi t/3\)), 2 sin(2\(\pi t/3\)) where the angles are in radians.
(b) 3 sin(40\(t\)), 4 cos(30\(t\) − 45) where the angles are in degrees.

3.7 Some rules involving trigonometric formulae

There are numerous equalities and rules that can be derived from our definitions of sin, cos and tan. The following are some of the more straightforward and useful:

\[
\begin{align*}
\cos(\pi/2 - \theta) &= \sin \theta \\
\sin^2 \theta + \cos^2 \theta &= 1 \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\sin(\alpha + \beta) &= \cos \alpha \sin \beta + \sin \alpha \cos \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta.
\end{align*}
\]

3.8 Derivatives and integrals of trigonometric expressions

Since the trigonometric functions have been introduced into this course for the sake of modelling cyclical systems, it should not be surprising that their derivatives are equally important since we are often required to find optimal values for certain functions. The basic rules of differentiation apply and the following results can be derived from first principles if necessary. Bear in mind that an integral can be regarded as the reverse of differentiation (for example, if \(d(\sin x)/dx = \cos x\) then \(\int \cos x \\(dx = \sin x)\)).

<table>
<thead>
<tr>
<th>Function (f(x))</th>
<th>Derivative (df(x)/dx)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin x)</td>
<td>(\cos x)</td>
</tr>
<tr>
<td>(\cos x)</td>
<td>(- \sin x)</td>
</tr>
<tr>
<td>(\tan x)</td>
<td>((\cos x)^{-2}), i.e. sec(^2) (x)</td>
</tr>
</tbody>
</table>

Activity 3.3

(a) Determine the differential of each of the following functions.
   i. \(\sin(2x)\)
   ii. \(x \cos 2x\)
   iii. \(3 \sin^2(4x)\)

(b) Determine the integral of each of the following functions.
3.9 Trigonometric series as expansions

For approximations and certain solution procedures it is often useful to expand the trigonometric functions (particularly sin and cos) in a power series:

\[
\sin x = x - x^3/3! + x^5/5! - x^7/7! + \cdots \\
\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots .
\]

These expansions are always valid in the sense that the right-hand side always converges to the left-hand side for every real value of \( x \).

3.10 Other trigonometric functions: reciprocals and inverse functions

The following definitions will prove useful when reading certain texts:

\[
\sec x = (\cos x)^{-1} \\
\cosec x = (\sin x)^{-1} \\
\cot x = (\tan x)^{-1}.
\]

If \( \sin x = y \) then \( x = \sin^{-1} y = \arcsin(y) \); if \( \cos x = y \) then \( x = \cos^{-1} y = \arccos(y) \) and if \( \tan x = y \) then \( x = \tan^{-1} y = \arctan(y) \).

3.11 Complex numbers

When solving quadratic functions of the form \( ax^2 + bx + c \) we know that the two solutions are

\[
x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a},
\]

which are real numbers so long as \( b^2 - 4ac \) is non-negative. When \( b^2 - 4ac < 0 \), however, we can still solve the quadratic if we introduce \( i = \sqrt{-1} \). Thus we have

\[
x = \frac{-b - i\sqrt{4ac - b^2}}{2a}, \quad \text{and} \quad x = \frac{-b + i\sqrt{4ac - b^2}}{2a}
\]
as the two ‘imaginary’ solutions.

We have thus created a new number system – this is where we might have a combination of a real number and an imaginary number. This mixed number system is called complex numbers. The complex number system consists of all expressions of the form \( a + ib \) where \( a \) and \( b \) are real numbers and \( i = \sqrt{-1} \), as defined earlier. \( a \) is called the real part of the complex number and \( ib \) is called the imaginary part. Complex numbers obey all the usual laws of algebra.
3.12 Conjugates

The (complex) conjugate of the complex number $z = a + ib$ is defined as $\bar{z} = a - ib$. We see that if a certain complex number is the solution to a quadratic equation then the conjugate complex number is the other solution.

3.13 The Argand diagram

Any single complex number $z = a + ib$ can be represented as a point on a two-dimensional (2D) graph where the axes are the real and imaginary parts of the complex number and the coordinates of $z$ are $(a, b)$ (see Figure 3.6). Thus, using our knowledge of trigonometry we can write $a = r \cos \theta$ and $b = r \sin \theta$ and hence

$$z = r(\cos \theta + i \sin \theta).$$

This is called the trigonometric form of $z$; $r = \sqrt{a^2 + b^2}$ is called the modulus of $z$ and is written $|z|$; the angle $\theta$ where $-\pi < \theta \leq \pi$ is called the argument of $z$ (arg $z$).

Figure 3.6: The Argand diagram.

3.14 De Moivre’s theorem

This states that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$  

The theorem also holds for negative as well as positive values of $n$. Furthermore

$$(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta.$$  

Example 3.1  Suppose we wish to find the real and imaginary parts of $z^n$ where $z = a + ib$.  

First we write \( z \) as
\[
(a^2 + b^2)^{0.5}(\cos \theta + i \sin \theta)
\]
where \( \theta = \arctan(b/a) \).
Hence \( z^n = (a^2 + b^2)^{n/2}(\cos n\theta + i \sin n\theta) \) i.e. the answer has a real part of \((a^2 + b^2)^{n/2} \cos n\theta\) and an imaginary part of \((a^2 + b^2)^{n/2} \sin n\theta\).

### 3.15 A link between exponential expansions, trigonometric functions and imaginary numbers

It can be shown that the exponential function \( e^x \) can be expanded as
\[
\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\]
for all real \( x \). In a similar fashion, if \( z \) is a complex number,
\[
\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots
\]
and when \( z = iu \) this becomes
\[
e^{iu} = 1 + iu + \frac{(iu)^2}{2!} + \frac{(iu)^3}{3!} + \frac{(iu)^4}{4!} + \cdots
\]
\[
= 1 + iu - \frac{u^2}{2!} - \frac{iu^3}{3!} + \frac{u^4}{4!} + \cdots
\]
\[
= \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right) + i \left(\frac{u}{3!} - \frac{u^3}{5!} - \cdots\right)
\]
\[
= \cos u + i \sin u.
\]

Hence we can write a complex number \( z \) in the form \( re^{i\theta} \) and using De Moivre’s theorem \( z^n = (e^{i\theta})^n = e^{i\theta n} \).

[You might note, as an aside, that \( e^{i\pi} = -1 \). Perhaps some of you will get the same sense of amazement as the author always does when he sees such an equation relating two irrational numbers \( e \) and \( \pi \) and the square root of minus one!]

### 3.16 Summary

This chapter has apparently been based on pure mathematics. However its importance becomes more obvious when the knowledge acquired is used in practical situations. We will return to trigonometric functions in Chapters 4 and 5.

The fairly extensive coverage of trigonometric functions in this chapter still leaves a lot of material uncovered.
3. Trigonometric functions and imaginary numbers

What you do not need to know

- The detailed integration problems concerning $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.
- Hyperbolic functions i.e. sinh, cosh and tanh. If they are completely meaningless to you then don’t worry!

3.17 Solutions to activities

3.1 (a) $\cos \left(\frac{5\pi}{3}\right) = 1/2$, $\sin \left(-\frac{\pi}{6}\right) = -1/2$, $\tan \left(\frac{7\pi}{3}\right) = \sqrt{3}$, $\sin \left(\frac{11\pi}{3}\right) = -\sqrt{3}/2$, $\cos \left(\frac{3\pi}{4}\right) = -1/\sqrt{2}$, $\sin \left(\frac{7\pi}{6}\right) = -1/2$, $\tan \left(\frac{7\pi}{4}\right) = -1$.

(b) $\cos(135) = -1/\sqrt{2}$, $\tan(-45) = -1$, $\sin(225) = -1/\sqrt{2}$, $\cos(-45) = 1/\sqrt{2}$, $\sin(300) = -\sqrt{3}/2$, $\tan(420) = \sqrt{3}$.

3.2 (a)
3.18 A reminder of your learning outcomes

By the end of this chapter, and having completed the Essential reading and activities, you should be able to:

- evaluate the values of trigonometric functions
- sketch graphs of the three main trigonometric functions and functions of them
- differentiate and integrate functions involving trigonometric functions
3. Trigonometric functions and imaginary numbers

- use series expansions of trigonometric functions and exponentials
- manipulate and use imaginary numbers
- use De Moivre’s theorem to interchange between complex numbers and trigonometric functions
- represent complex numbers on an Argand diagram
- understand the meaning and usefulness of complex conjugates.

3.19 Sample examination questions

(Please note that each of these sample questions is only part of a full examination question.)

1. The rate of sales, \( \frac{dS}{dt} \), of a product in a market with cyclical demand is modelled by

\[
\frac{dS}{dt} = 500 \left( 1 + \sin \frac{\pi t}{10} \right)
\]

where \( t \) is measured in weeks.

Determine the total volume of sales of the new product within the first four weeks using:

(a) direct integration

(b) series expansion of the \( \sin \) function up to and including terms in \( t^5 \).

[Note: You may assume that \( \pi = 3.1416 \).]

(6 marks)

(6 marks)

2. Find the real and imaginary parts of

(a) \( \frac{2 + 3i}{3 + 2i} \)

(b) \( \frac{1}{i^5} \)

(c) \( \log_e \left( \frac{1}{2} (\sqrt{3} + i) \right) \)

(d) \( (4 + 3i)e^{i\pi/3} \).

(12 marks)

3. You are given the complex numbers \( z = 2 - 3i \) and \( w = 1 + 4i \). Find the real and imaginary parts of

(a) \( z - w \)

(b) \( zw \)

(c) \( z/w \)

(d) \( z^8 \).

(10 marks)
3.20 Guidance on answering the Sample examination questions

3. (a) Expand $e^{\sin x}$ as a series up to terms in $x^4$ and hence evaluate

$$\int_0^{\pi/3} e^{\sin x} \, dx.$$  

(7 marks)

[Note: You may assume that $\pi = 3.1416$.]

(b) Find the real and imaginary parts of

i. $(4 - 3i)/(3 + 2i)$

ii. $\log_e \left( \frac{1}{\sqrt{2}}(1 - i) \right)$

and draw an Argand diagram for your answer to (a).

(7 marks)

3.20 Guidance on answering the Sample examination questions

1. (a) We have

$$S = \int_0^4 500 \left( 1 + \sin \frac{\pi t}{10} \right) \, dt$$

$$= 500 \left[ t - \frac{10}{\pi} \cos \frac{\pi t}{10} \right]_0$$

$$= 500 \left[ 4 + \frac{10}{\pi} \left( 1 - \cos \frac{4\pi}{10} \right) \right]$$

$$= 500[4 + 3.1831(1 - 0.3090)]$$

$$= 500[6.19947]$$

$$= 3099.73.$$  

[Note: The answer can be left as a function of $\cos$ when only basic calculators are permitted.]

(b) We have

$$\sin \frac{\pi t}{10} = \frac{\pi t}{10} - \frac{1}{3!} \left( \frac{\pi t}{10} \right)^3 + \frac{1}{5!} \left( \frac{\pi t}{10} \right)^5 - \cdots.$$  

Therefore,

$$S = 500 \int_0^4 \left( 1 + \frac{\pi t}{10} - \frac{\pi^3 t^3}{6000} + \frac{\pi^5 t^5}{1200000} - \cdots \right) \, dt$$

$$\approx 500 \left[ t + \frac{\pi t^2}{20} - \frac{\pi^3 t^4}{24000} + \frac{\pi^5 t^6}{7200000} \right]_0$$

$$\approx 500[4 + 2.5133 - 0.3307 + 0.0174]$$

$$\approx 500[6.2]$$

$$= 3100.$$
3. Trigonometric functions and imaginary numbers

2. (a) We have
\[
\frac{2 + 3i}{3 + 2i} = \left( \frac{2 + 3i}{3 + 2i} \right) \cdot \left( \frac{3 - 2i}{3 - 2i} \right) = \frac{6 + 9i - 4i + 6}{9 - 4i^2} = \frac{12 + 5i}{13} = \frac{12}{13} + \frac{5}{13}i.
\]

(b) We have
\[
\frac{1}{i^5} = \frac{1}{i^2i^3} = \frac{1}{(-1)(-1)i} = \frac{i}{1} = \frac{i}{-1} = -i.
\]

(c) We have
\[
\log e \left( \frac{1}{2} (\sqrt{3} + i) \right) = \log e \left( \sin \frac{\pi}{3} + i \cos \frac{\pi}{3} \right),
\]

or, more usefully,
\[
\log e \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \log e (e^{i\pi/6}) = \frac{i\pi}{6}.
\]

(d) We have
\[
(4 + 3i)e^{i\pi/3} = (4 + 3i) \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = (4 + 3i) \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = \left( \frac{4 - 3\sqrt{3}}{2} \right) + i \left( \frac{3 + 4\sqrt{3}}{2} \right).
\]

3. (a) We have
\[
z - w = (2 + 3i) - (1 - 4i) = (1 + 7i).
\]

(b) We have
\[
zw = (2 + 3i)(1 - 4i) = 2 - 5i - 12i^2 = 14 - 5i.
\]

(c) we have
\[
\frac{z}{w} = \frac{2 + 3i}{1 - 4i} = \frac{(2 + 3i)(1 + 4i)}{(1 - 4i)(1 + 4i)} = \frac{2 + 11i - 12}{1 + 16} = \frac{-10}{17} + \frac{11}{17}i.
\]

(d) We have
\[
z^7 = (2 + 3i)^7 = \sqrt{137} \cos \theta + i \sin \theta^7
\]
where \(\theta = \tan^{-1}(3/2) = [56^\circ 31']\), so
\[
z^7 = \sqrt{137} \cos 7\theta + i \sin 7\theta \approx 7921.4(0.8274 + i0.5616) = 6554 + 4449i.
\]

Note: You may omit the parts in square brackets ‘[ ]’ if trigonometric functions are not permitted by the current calculator regulations.

4. (a) We have
\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]
so

\[
e^{\sin x} = 1 + \frac{\sin x}{1!} + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \cdots
\]

\[
= 1 + x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots + \frac{1}{2} \left( x^2 + \frac{x^6}{36} - \frac{x^4}{3} + \frac{x^6}{60} - \cdots \right) \nonumber
\]

\[
+ \frac{1}{6} \left( x^3 - \frac{x^5}{2} + \cdots \right) + \frac{1}{24} x^4 + \cdots \nonumber
\]

\[
= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots . \nonumber
\]

Hence,

\[
\int_0^{\pi/3} e^{\sin x} \, dx \approx \left[ x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40} \right]_0^{\pi/3} \nonumber
\]

\[
= 1.04720 + 0.54831 + 0.19140 - 0.03148 \nonumber
\]

\[
= 1.7554 . \nonumber
\]

(b) i. We have

\[
\frac{4 - 3i}{3 + 2i} = \frac{(4 - 3i)(3 - 2i)}{(3 + 2i)(3 - 2i)} = \frac{12 - 17i - 6}{9 + 4} = \frac{6}{13} - \frac{17}{13}i . \nonumber
\]

ii. We have

\[
\log_e \left( \frac{1}{\sqrt{2}}(1 - i) \right) = \log_e (e^{iu}) \nonumber
\]

where \( \cos u = 1/\sqrt{2} \), \( \sin u = 1/\sqrt{2} \) i.e. \( u = \frac{7\pi}{4} \). Hence

\[
\log_e \left( \frac{1}{\sqrt{2}}(1 - i) \right) = iu = \frac{7\pi}{4}i . \nonumber
\]
3. Trigonometric functions and imaginary numbers