Advanced mathematical analysis
M. Anthony
MT3041, 2790041
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Undergraduate study in Economics, Management, Finance and the Social Sciences

This is an extract from a subject guide for an undergraduate course offered as part of the University of London International Programmes in Economics, Management, Finance and the Social Sciences. Materials for these programmes are developed by academics at the London School of Economics and Political Science (LSE).

For more information, see: www.londoninternational.ac.uk
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Chapter 1
Introduction

In this very brief introduction, I aim to give you an idea of the nature of this course and to advise on how best to approach it. I also give general information about the contents and use of this subject guide, and on recommended reading and how to use the textbooks.

1.1 This course

1.1.1 Relationship to previous mathematics courses

If you are taking this course as part of a BSc degree you will already have taken the pre-requisite Mathematics course 116 Abstract Mathematics. In 116 Abstract Mathematics you will have learned about the fundamentals of mathematical analysis: in particular, you will have studied sequences and functions, limits and continuity. This course continues study in mathematical analysis, extending the material in 116 Abstract Mathematics. The emphasis is on series, functions and sequences in \( n \)-dimensional real space, and we shall also study the general (and unifying) concept of a metric space.

After studying this course, you should be equipped with an understanding of concepts (such as continuity and compactness) which are central not only to further mathematical courses, but to applications of mathematics in economics and other areas. For example, as we shall see, compactness is a very important idea in optimisation. The course will also enable you to set the real analysis you previously encountered in a larger context, to see that there is a ‘bigger picture’. More generally, a course of this nature, with the emphasis on abstract reasoning and proof, will help you to think in an analytical way, and be able to formulate mathematical arguments in a precise, logical manner.

1.1.2 Aims

The half course is designed to enable you to:

- develop further your ability to think in a critical manner
- formulate and develop mathematical arguments in a logical manner
- improve your skill in acquiring new understanding and experience
- acquire an understanding of advanced mathematical analysis.
1. Introduction

1.1.3 Learning outcomes

At the end of this half course and having completed the Essential reading and activities, you should:

- have a good knowledge of mathematical concepts in real analysis
- be able to use formal notation correctly and in connection with precise statements in English
- be able to demonstrate the ability to solve unseen mathematical problems in real analysis
- be able to prove statements and to formulate precise mathematical arguments.

1.1.4 Topics covered

Descriptions of topics to be covered appear in the relevant chapters.

We study the formal mathematical theory of:

- series of real numbers;
- series and sequences in $n$-dimensional real space $\mathbb{R}^n$;
- limits and continuity of functions mapping between $\mathbb{R}^m$ and $\mathbb{R}^n$;
- differentiation;
- the topology of $\mathbb{R}^n$;
- metric spaces;
- uniform convergence of sequences of functions;

Not all chapters of the guide are the same length. It should not be thought that you should spend the same amount of time on each chapter. I will not try to specify precisely how much relative time should be spent on each: that will vary from person to person and I do not want to be prescriptive. However, as a very rough guide (bearing in mind that this must vary from individual to individual), I would suggest that the percentages of time spent on each chapter are something along the lines suggested in Table 1.1.

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Table 1.1: Suggested allocation of time on the various chapters

1.2 Recommended reading

You are advised to read books. This subject guide is just a guide, and is not a textbook. There are many books that would be useful for this course, since Mathematical Analysis is a major component of most university-level mathematics degree programmes. There
is no single book that corresponds exactly to this course as it is treated here, but there are many books that are useful for parts of it.

The following books are recommended.


None of these books covers all of the topics in this course. ‘Yet Another Introduction to Analysis’ was highly recommended for 116 Abstract Mathematics, so you may already have it. It will be useful for Chapters 2 and 4, and it will also be useful for revising the material you will need to know from 116 Abstract Mathematics. The Binmore book will be useful for Chapters 2, 3 and 4. Brannan’s book will be useful for Chapters 2 and 4. The book by Bartle and Sherbert will be useful for Chapters 2, 4 and 7, and will also be of some use for Chapters 5 and 6. Of more use for Chapters 5 and 6 are the ‘Metric Spaces’ book of Bryant and the Sutherland book. Note, however, that most of the Sutherland book covers more advanced topics than this subject, and the Bryant Metric Spaces book takes a slightly different approach from that taken here.

Detailed reading references in this subject guide refer to the editions of the set textbooks listed above. New editions of one or more of these textbooks may have been published by the time you study this course. You can use a more recent edition of any of the books; use the detailed chapter and section headings and the index to identify relevant readings. Also check the VLE regularly for updated guidance on readings.

Please note that as long as you read the Essential reading you are then free to read around the subject area in any text, paper or online resource. To help you read extensively, you have free access to the virtual learning environment (VLE) and University of London Online Library (see below).

### 1.3 Online study resources

In addition to the subject guide and the Essential reading, it is crucial that you take advantage of the study resources that are available online for this course, including the virtual learning environment (VLE) and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at:

http://my.londoninternational.ac.uk
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You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged in to the Student Portal in order to register! As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you forget your login details at any point, please email uolia.support@london.ac.uk quoting your student number.

1.3.1 The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of subject material.
- Electronic study materials: The printed materials that you receive from the University of London are available to download, including updated reading lists and references.
- Past examination papers and Examiners’ commentaries: These provide advice on how each examination question might best be answered.
- A student discussion forum: This is an open space for you to discuss interests and experiences, seek support from your peers, work collaboratively to solve problems and discuss subject material.
- Videos: There are recorded academic introductions to the subject, interviews and debates and, for some courses, audio-visual tutorials and conclusions.
- Recorded lectures: For some courses, where appropriate, the sessions from previous years’ Study Weekends have been recorded and made available.
- Study skills: Expert advice on preparing for examinations and developing your digital literacy skills.
- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

1.3.2 Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login:

http://tinyurl.com/ollathens

The easiest way to locate relevant content and journal articles in the Online Library is to use the Summon search engine.
1.4 Using the guide

The Sample questions at the end of the chapters of this guide are a very useful resource. You should try them once you think you have mastered a particular chapter. Really try them: don’t just simply read the solutions provided. Make a serious attempt before consulting the solutions. Note that the solutions are often just sketch solutions, to indicate to you how to answer the questions. However, in the examination, you must show all your reasoning. It is vital that you develop and enhance your problem-solving skills and the only way to do this is to try lots of examples.

At certain points I have tried hard to emphasise certain pitfalls that students can fall into in their understanding of key concepts and I have indicated these by the use of the word ‘WARNING!’. Please pay particular attention to these: the concepts and results in this course can sometimes be quite subtle and misunderstandings can easily arise.

You will need to know, and be able to reproduce, some of the proofs of standard results, as indicated in the learning outcomes. I have also included in this guide some other proofs that you will not be expected to reproduce. I have done so in order to help you understand the material better.

Finally, we often use the symbol □ to denote the end of a proof, where we have finished explaining why a particular result is true. This is just to make it clear where the proof ends and the following text begins.

1.5 Examination advice

Important: the information and advice given here are based on the examination structure used at the time this guide was written. Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current Regulations for relevant information about the examination, and the virtual learning environment (VLE) where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and Examiners commentaries for the course which give advice on how each question might best be answered.

Two Sample examination papers are given as an appendix to this guide. There are no optional topics in this course: you should study them all. The examination paper will provide some element of choice as to which questions you attempt; see the Sample
1. Introduction

examination papers at the end of the guide for an indication of the structure of the examination paper.

Please do not assume that the questions in a real examination will necessarily be very similar to these sample questions. An examination is designed (by definition) to test you. You will get examination questions unlike questions in this guide and each year there will be examination questions different from those in previous years. The whole point of examining is to see whether you can apply knowledge in familiar and unfamiliar settings. For this reason, it is important that you try as many examples as possible, from the guide and from the textbooks. This is not so that you can cover any possible type of question the examiners can think of! It’s so that you get used to confronting unfamiliar questions, grappling with them, and finally coming up with the solution.

Do not panic if you cannot completely solve an examination question. There are many marks to be awarded for using the correct approach or method.

1.6 The use of calculators

You will not be permitted to use calculators of any type in the examination. This is not something that you should panic about: the examiners are interested in assessing that you understand the key concepts, ideas, methods and techniques, and will set questions which do not require the use of a calculator.
Chapter 2
Series of real numbers

Reading
- Bryant, Victor. *Yet Another Introduction to Analysis*. Chapter 2.
- Brannan, David. *A First Course in Mathematical Analysis*. Chapter 3.

2.1 Introduction

The first main topic of the course is *series*. This chapter looks at how one can formalise and deal properly with infinite sums. A key question is whether an infinite sum exists (that is, whether a series converges).

To understand series, we need to understand *sequences*. We start, therefore, by racing through some of the results you should know already from *116 Abstract Mathematics* about sequences. (The discussion of this background material is deliberately brief: you can find more information in *116 Abstract Mathematics* and the reading cited.)

2.2 Revision: sequences

Formally, a *sequence* is a function $f$ from $\mathbb{N}$ to $\mathbb{R}$. We call $f(n)$ the $n^{th}$ term of the sequence and we often denote the sequence by $(f(n))_{n=1}^\infty$ or simply $(f(n))$. Informally a sequence is an infinite list of real numbers, one for each positive integer; for example,

$$a_1, a_2, a_3, \ldots$$

We denote it $(a_n)_{n=1}^\infty$ or $(a_n)$ (or indeed, $(a_r), (a_i)$ etc.). Then we call $a_n$ the $n^{th}$ term of this sequence.

A sequence may be defined by giving an explicit formula for the $n^{th}$ term. For example the formula $a_n = \frac{1}{n}$ defines the sequence whose value at the positive integer $n$ is $\frac{1}{n}$.

A sequence may also be defined inductively. For instance, we might have

$$a_1 = 1, \quad a_{n+1} = \frac{a_n}{2} + \frac{3}{2a_n} \quad (n \geq 1).$$
2. Series of real numbers

2.2.1 Limits of sequences

The formal definition of a limit is as follows.

**Definition 2.1 (Finite limit of a sequence)** The sequence \( (x_n) \) is said to tend to the (finite) limit \( L \) if for all \( \epsilon > 0 \), there is an integer \( N \) such that for all \( n > N \) we have \( |x_n - L| < \epsilon \). That is, \( (x_n) \) tends to \( L \) if

\[
\forall \epsilon > 0, \exists N \text{ such that } n > N \implies |x_n - L| < \epsilon.
\]

We write \( x_n \to L \text{ as } n \to \infty \)

or

\[
\lim_{n \to \infty} x_n = L,
\]

and say that \( x_n \) tends to \( L \) as \( n \) tends to \( \infty \).

Any sequence which tends to a finite limit is said to be convergent.

**Theorem 2.1** A sequence has no more than one limit.

2.2.2 Boundedness and monotonicity

**Definition 2.2 (Bounded sequences)** A sequence is bounded above/bounded/bounded below if the set

\[
S = \{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}
\]

is bounded above/bounded/bounded below.

**Theorem 2.2** Any convergent sequence is bounded. That is,

\[
\text{convergent} \implies \text{bounded}.
\]

**Definition 2.3 (Monotonic sequences)** A sequence \( (a_n) \) is increasing (decreasing) if for all \( n \), \( a_{n+1} \geq a_n \) (\( a_{n+1} \leq a_n \)). A sequence is monotonic if it is either increasing or decreasing.

**Theorem 2.3** An increasing (decreasing) sequence which is bounded above (below) is convergent. That is,

\[
\text{bounded} + \text{monotonic} \implies \text{convergent}.
\]

2.2.3 The Algebra of Limits

**Theorem 2.4 (`Algebra of limits` results)** Suppose \( (a_n) \) and \( (b_n) \) are convergent sequences with

\[
a_n \to a, \ b_n \to b, \ as \ n \to \infty.
\]
Then as \( n \to \infty \),

1. \( C a_n \to C a \) for any real number \( C \),
2. \( |a_n| \to |a| \),
3. \( a_n + b_n \to a + b \),
4. \( a_n b_n \to ab \),
5. If \( b_n \neq 0 \ \forall \ n \) and if \( b \neq 0 \), then \( \frac{1}{b_n} \to \frac{1}{b} \),
6. \( a_n^k \to a^k \) for any positive integer \( k \).

Also, note that if \( a_n \geq 0 \) for all \( n \), then \( a \geq 0 \).

### 2.2.4 Subsequences

The formal definition of a subsequence is as follows.

**Definition 2.4 (Subsequence)** Let \((a_n)\) be a sequence and let \( f \) be a strictly increasing function from \( \mathbb{N} \) to \( \mathbb{N} \). The sequence \((a_{f(n)})\) is called a *subsequence* of the sequence \((a_n)\).

[Note that a function is strictly increasing if \( f(n + 1) > f(n) \) for all positive \( n \).]

Often we will use shorthand to denote a sequence. For example, for a sequence \( a_1, a_2, a_3, a_4, \ldots \) we may say that we wish to choose a subsequence

\[ a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4}, \ldots \]

We have written the increasing function \( f \) explicitly in terms of its value, by saying exactly which terms of the original sequence to take; that is, we take terms \( k_1, k_2, k_3 \) and so on. Note that this will be just for notation’s sake and in these cases the underlying function has not disappeared; in fact the increasing function here is given by \( f(i) = k_i \) for \( i = 1, 2, 3, \ldots \).

Another notation, sometimes useful, is to let \( A = \{ f(n) : n \in \mathbb{N} \} \) and denote the subsequence \((a_{f(n)})\) by \((a_n)_{n \in A}\). For any infinite set \( A \) of natural numbers, \((a_n)_{n \in A}\) is a subsequence of \((a_n)\). This notation is particularly useful if we ever have to form a subsequence of a subsequence. For example, if \( B \) and \( A \) are infinite subsets of \( \mathbb{N} \), with \( B \subseteq A \), then \((a_n)_{n \in B}\) is a subsequence of \((a_n)_{n \in A}\), which in turn is a subsequence of \((a_n)\). This approach avoids the need for double and triple subscripts.

One result which is easy to show is:

**Theorem 2.5** Let \((a_n)\) be a sequence which tends to a limit \( L \). Then any subsequence also tends to the limit \( L \).

Another result, less easy to prove, but useful is the following.
Theorem 2.6  Every sequence has a monotonic subsequence.

We obtain from this as a corollary the following famous result.

Theorem 2.7 (Bolzano-Weierstrass Theorem) Every bounded real sequence has a convergent subsequence.

(The proof is immediate. Let \((a_n)\) be a bounded sequence; by the preceding theorem, it has a monotonic subsequence \((a_{f(n)})\). This subsequence is then also bounded and we have seen that bounded monotonic subsequences are convergent.)

2.3 Series

The previous material is revision from 116 Abstract Mathematics. Now we start on new material.

In this part of the course, we will be concerned with how one can formalise the idea of summing an infinite list of numbers

\[ a_1 + a_2 + a_3 + \ldots \]

As you would expect, this will once again involve the notion of a limit. We begin with a basic definition:

Definition 2.5  Let \((a_n)_{n=1}^{\infty}\) be a sequence. For each \(n \geq 1\), let

\[ s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k. \]

The series with \(n\)th term \(a_n\), denoted \(\sum a_n\), is, formally, the sequence \((s_n)\). We call \(a_n\) the \(n\)th term of the series and \(s_n\) is called the \(n\)th partial sum of the series.

Although we denote a series by the notation \(\sum a_n\), some textbooks use the notation \(\sum_{n=1}^{\infty} a_n\). We shall reserve that notation for something different, as I will explain below.

Note that, as far as we are concerned, a series involves an infinite list of numbers. We do not discuss ‘finite’ series, since there are no convergence issues there.

Let’s consider an example.

Example 2.1  \(\sum (-1)^n\).

The \(n\)th term is \((-1)^n\) and the \(n\)th partial sum \(s_n\) is \(-1\) if \(n\) is odd and \(0\) if \(n\) is even.

Activity 2.1  What is the \(n\)th partial sum of the series \(\sum n^2\)?

2.4 Convergence of series

Definition 2.6  Let \(\sum a_n\) be a series. If the sequence \((s_n)\) of partial sums converges to \(L\) (finite), then we say that the series converges to \(L\), or has sum \(L\). If \((s_n)\) diverges, then we say that \(\sum a_n\) diverges.
If a series $\sum a_n$ converges to $L$, we write $\sum_{n=1}^{\infty} a_n = L$. (Here is what we use the notation $\sum_{n=1}^{\infty} a_n$ for: we use it to mean the sum of the series, when the series converges.)

**Warning!**

Be clear in your mind that the convergence of a series $\sum a_n$ is about convergence of the sequence of partial sums $(s_n)$. It is not about convergence of $(a_n)$.

**Theorem 2.8** If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

**Proof**

Because the series converges, there is some number $L$ such that $s_n \to L$ as $n \to \infty$. Now, if $s_n \to L$, we also have $s_{n-1} \to L$. It follows that $s_n - s_{n-1} \to L - L = 0$. But what is $s_n - s_{n-1}$? Well, it is precisely $a_n$. So we have $a_n \to 0$ as $n \to \infty$.

Note that, here, we have used the symbol ‘□’ to denote the end of the proof. This is a convenient way of indicating when a proof is over and the main text continues.

**Warning!**

You should be aware that the converse of this result is false: $a_n$ tending to 0 does not necessarily mean that the series $\sum a_n$ converges. We shall see specific examples shortly of series in which $a_n \to 0$ and yet the series diverges. Finding sufficient conditions for a series to converge is the main aim in what follows, and it’s not easy. Life would be simple if it were the case that $\sum a_n$ converges if and only if $a_n \to 0$, but it isn’t so. What this means is that you should never find yourself saying or writing ‘$a_n \to 0$ and therefore the series converges.’ It is never possible to conclude that a series converges just from the fact that $a_n \to 0$.

However, the result is useful, sometimes, for proving that a series does not converge, for it is equivalent to the following result. (This is the contrapositive of Theorem 2.8.)

**Theorem 2.9** If $a_n$ does not tend to 0 as $n \to \infty$, then the series $\sum a_n$ diverges.

**Activity 2.4** Prove that the series $\sum (-1)^n$ diverges.

You should not think that if a series diverges, then it must be the case that $s_n \to \infty$. This will turn out to be the case if all terms $a_n$ of the series are non-negative, but it is not true in general. For example, $\sum (-1)^n$ diverges, but we do not have $s_n \to \infty$.

The following result establishes that, for a series with non-negative terms (a ‘non-negative series’), either the series converges, or the partial sums tend to infinity.

**Theorem 2.10** Suppose that $\sum a_n$ is a series in which $a_n \geq 0$ for all $n$. If the series diverges, then the $n$th partial sum, $s_n$, is such that $s_n \to \infty$ as $n \to \infty$. 
2. Series of real numbers

Proof
We start by observing that the partial sums of a non-negative series form an increasing sequence. This is because \( s_{n+1} = s_n + a_{n+1} \geq s_n \), since \( a_{n+1} \geq 0 \). If the sequence \( (s_n) \) was bounded above, then it would converge, because an increasing sequence that is bounded above converges. So it must be the case that \( (s_n) \) is not bounded above. So, for each \( K \) there is \( N \) such that \( s_N > K \). But then we have that, for all \( n \geq N \), \( s_n \geq s_N > K \). This shows that \( s_n \rightarrow \infty \).

Comment. You might well see that the proof of Theorem 2.10 shows us something else: namely, that if the partial sums of a non-negative series form a bounded sequence, then the series converges. This is because the partial sums are an increasing sequence. So, for non-negative series, the question of convergence becomes one about whether the partial sums are bounded: explicitly, for a non-negative series, the series converges if and only if the partial sums are bounded.

Warning!
Theorem 2.10 is only true for series with non-negative terms. If that’s not clear, consider \( \sum (-1)^n \). The partial sums of this series are bounded (for they are all either \(-1\) or \(0\)), but the series does not converge.

2.5 Special series

Here’s a classic example that you’ll be familiar with.

Theorem 2.11 (Geometric Series) Let \( a, r \in \mathbb{R} \). Then

1. \( \sum ar^{n-1} \) converges to \( \frac{a}{1-r} \) if \( |r| < 1 \).
2. \( \sum ar^{n-1} \) diverges if \( |r| \geq 1 \).

Proof
The partial sum of the geometric series is \( s_n = \frac{a(1-r^n)}{1-r} \) if \( r \neq 1 \). If \( |r| < 1 \) then this converges to the limit \( \frac{a}{1-r} \) because in that case \( r^n \rightarrow 0 \). If \( |r| > 1 \), it does not converge because \( r^n \rightarrow \infty \) when \( r > 1 \) and \( r^n \) oscillates unboundedly when \( r < -1 \). The only remaining cases are when \( r = 1 \) and \( r = -1 \). When \( r = 1 \), \( s_n \) is simply \( na \) and this tends to infinity and therefore does not converge. When \( r = -1 \), the partial sums are alternately \( a \) and \( 0 \), and this sequence of numbers does not converge. \( \square \)

The following result is extremely useful. The series \( \sum \frac{1}{n} \) is so special that it has a special name: the harmonic series.

Theorem 2.12 The harmonic series \( \sum \frac{1}{n} \) diverges.
Proof
Let $s_n$ denote the partial sum. Then
\[
s_{2n} - s_n = (a_1 + a_2 + \cdots + a_{2n}) - (a_1 + a_2 + \cdots + a_n)
= a_{n+1} + a_{n+2} + \cdots + a_{2n}
= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}
\geq \frac{n}{2n}
= \frac{1}{2}.
\]
Now, if the series converges, we should have $s_n \to L$ for some $L$. Then we’d also have $s_{2n} \to L$ and hence $s_{2n} - s_n \to L - L = 0$. But this cannot be, because we’ve shown that $s_{2n} - s_n \geq 1/2$ for all $n$. Hence the series diverges.

The following more general result is going to be very useful to us. The second part of its proof is difficult and you would not be expected to reproduce it in an examination.

**Theorem 2.13** The series $\sum \frac{1}{n^s}$ diverges if $s \leq 1$ and converges if $s > 1$.

**Proof**

The first part of the proof, in which we suppose $s \leq 1$, is similar to the proof of divergence of the harmonic series. (It can alternatively be proved by using Theorem 2.10 together with Theorem 2.12 and the fact that $1/n^s \geq 1/n$ if $s \leq 1$.) If $s_n$ denotes the partial sum, then we have
\[
s_{2n} - s_n = (a_1 + a_2 + \cdots + a_{2n}) - (a_1 + a_2 + \cdots + a_n)
= a_{n+1} + a_{n+2} + \cdots + a_{2n}
= \frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \cdots + \frac{1}{(2n)^s}
\geq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}
\geq \frac{n}{2n}
= \frac{1}{2}.
\]
As in the proof above, this shows that we cannot have $(s_n)$ converging and so the series diverges.

Now we prove that the series converges if $s > 1$.

Let $s_n$ denote the $n$th partial sum of the series. To prove that the series converges is, by definition, to prove that the sequence $(s_n)$ converges. Since the terms of the series are positive, the sequence $(s_n)$ is increasing, so to establish convergence it is sufficient to show that it is bounded above. (Remember that an increasing sequence that is bounded above must converge.)
2. Series of real numbers

Obviously $s_n \leq s_{2^n-1}$, although you might wonder why we make this observation. (You’ll see...).

Now,

$$s_{2^n-1} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{(2^n - 1)^n}$$

$$= 1 + \left(\frac{1}{2^n} + \frac{1}{3^n}\right) + \left(\frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n}\right) + \cdots$$

$$\cdots + \left(\frac{1}{(2^n-1)^n} + \frac{1}{(2^n-1+1)^n} + \cdots + \frac{1}{(2^n-1)^n}\right).$$

What we’ve done here is simply group the terms together. This does not change the value of the expression. We have taken the first term, then the next 2 together, then the next 4, then the next 8, and so on, until the last group, which is of size $2^n-1$. (Note that $1 + 2 + \cdots + 2^{n-1}$ does indeed equal $2^n - 1$.) The reason for doing this is that we can now bound this expression by noting that in each group, the largest term is the first in that group, so the value of each bracketed quantity is no more than the number of terms inside the brackets multiplied by the first of the terms. So,

$$s_{2^n-1} \leq 1 + 2 \frac{1}{2^n} + 4 \frac{1}{4^n} + 8 \frac{1}{8^n} + \cdots + 2^{n-1} \frac{1}{(2^n - 1)^n}$$

$$= 1 + \frac{1}{2^{n-1}} + \frac{1}{4^{n-1}} + \frac{1}{8^{n-1}} + \cdots + \frac{1}{(2^n-1)^{n-1}}$$

$$= 1 + \frac{1}{2^{n-1}} + \frac{1}{(2^{n-1})^2} + \frac{1}{(2^{n-1})^3} + \cdots + \frac{1}{(2^{n-1})^{n-1}}.$$

where we have just used the fact that $(2^i)^{n-1} = (2^{n-1})^i$. Now, consider the geometric series $\sum \frac{1}{(2^{n-1})^{n-1}}$. This has common ratio $1/2^{n-1}$ which is positive and less than 1, so the series converges. In fact, its sum is

$$L = \frac{1}{1 - (1/2^{n-1})}.$$

But the most important fact is that its partial sums are bounded (all are less than $L$). The calculation above shows that if $t_n$ is the $n$th partial sum of this geometric series, then $s_{2^n-1} \leq t_n$. It follows that $s_{2^n-1} \leq L$ and hence, as required, we have shown that the partial sums $s_{2^n-1}$ (and hence $s_n$) are bounded. This finishes the proof.

\[\square\]

**Activity 2.6** Prove that if $s \leq 1$ then $\sum 1/n^s$ diverges, by using Theorem 2.10 together with Theorem 2.12 and the fact that $1/n^s \geq 1/n$ if $s \leq 1$.

There also exist some ‘Algebra of Limits’ results which can be proved directly from the corresponding results for sequences:
2.6. Some useful tests for non-negative series

Theorem 2.14 Suppose $\sum a_n$ and $\sum b_n$ converge, and that $\sum_{n=1}^{\infty} a_n = L$ and $\sum_{n=1}^{\infty} b_n = M$. Then, for any real number $c$, the series $\sum (a_n + b_n)$ and $\sum c a_n$ converge, and $\sum_{n=1}^{\infty} (a_n + b_n) = L + M$ and $\sum_{n=1}^{\infty} c a_n = cL$.

But ... note that the same does not hold for products. For example, if $a_n = (-1)^n / \sqrt{n}$, then, as we will shortly see, $\sum a_n$ converges. However, $\sum (a_n \times a_n)$ diverges. This latter series is simply $\sum \frac{1}{n}$, the harmonic series.

2.6 Some useful tests for non-negative series

A series is non-negative if all its terms are non-negative. (Later we look at series which have some negative terms, but it’s easiest at the moment to stick to non-negative series.) The aim now is to develop a range of tests for convergence.

2.6.1 Comparison test

First, we have the Comparison Test.

Theorem 2.15 (Comparison Test) Let $(a_n), (b_n)$ be non-negative sequences such that $a_n \leq b_n$ for all $n$. Then

1. If $\sum b_n$ converges, then $\sum a_n$ does also, and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof

The key observation is that if $s_n$ and $t_n$ are, respectively, the $n$th partial sums of $\sum a_n$ and $\sum b_n$, then $s_n \leq t_n$. Suppose that $\sum b_n$ converges. This means precisely that the series $(t_n)$ converges (by the definition of convergence of a series). So the sequence $(t_n)$ is certainly bounded above. Now, $(s_n)$ is an increasing sequence and since $s_n \leq t_n$ for all $n$, $(s_n)$ is bounded above too. So, as an increasing sequence which is bounded above, it converges. Furthermore,

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n \leq \lim_{n \to \infty} t_n = \sum_{n=1}^{\infty} b_n.$$

Suppose, now, that $\sum a_n$ diverges. By Theorem 2.10, $s_n \to \infty$. So, $t_n \to \infty$ since $t_n \geq s_n$. Hence $(t_n)$ diverges and (by definition) $\sum b_n$ diverges.

When using the Comparison Test, it’s important to use it in the right direction. Suppose, for example, you want to use it to show that $\sum a_n$ converges. Then you need to find a series $\sum b_n$ that you know converges and which satisfies $0 \leq a_n \leq b_n$ for all $n$. If you wanted to use it to show that a series $\sum c_n$ diverges, you need a divergent series $\sum d_n$ with $c_n \geq d_n$.

The Comparison Test can be weakened slightly as follows. (Here, what we’ve done is replace ‘for all $n$’ with ‘for all sufficiently large $n$’.)
2. Series of real numbers

**Theorem 2.16** Let \((a_n), (b_n)\) be non-negative sequences such that there is some \(N\) such that \(a_n \leq b_n\) for all \(n \geq N\). Then

1. If \(\sum b_n\) converges, then \(\sum a_n\) does also.
2. If \(\sum a_n\) diverges, then \(\sum b_n\) diverges.

**Proof**
The key observation in the proof of the previous version of the Comparison Test was that, using the same notation, \(s_n \leq t_n\). That is no longer necessarily true in this case. However, it is true that there will be some constant \(M\) such that \(s_n \leq t_n + M\) for all \(n \geq N\). For,

\[
t_n - s_n = \sum_{i=1}^{n} (b_n - a_n) = \sum_{i=1}^{N-1} (b_n - a_n) + \sum_{i=N}^{n} (b_n - a_n).
\]

Now, let \(M = \sum_{i=1}^{n} (b_n - a_n)\). Then, noting that \(\sum_{i=N}^{n} (b_n - a_n) \geq 0\) because \(b_n \geq a_n\) for \(n \geq N\), we see that

\[
t_n - s_n \geq M + 0 = M.
\]

Now the proof is very similar to the one before. Suppose that \(\sum b_n\) converges. Then \((t_n)\) converges and so it is bounded above. The sequence \((s_n)\) is increasing and, since \(s_n \leq t_n + M\) for all \(n \geq N\), \((s_n)\) is bounded above. So, as an increasing sequence which is bounded above, it converges. Suppose, now, that \(\sum a_n\) diverges. By Theorem 2.10, \(s_n \to \infty\). So, \(t_n \to \infty\) since \(t_n \geq s_n - M\). Hence \((t_n)\) diverges and (by definition) \(\sum b_n\) diverges.

**Example 2.2** Consider

\[
\sum \frac{n^2 + 1}{n^5 + n + 1}.
\]

The \(n\)th term here behaves like \(1/n^3\), because the dominant term on the numerator is \(n^2\) and the dominant term in the denominator is \(n^5\). But this needs to be made precise. We can formally compare the series with \(\sum 1/n^3\) by noting that

\[
\frac{n^2 + 1}{n^5 + n + 1} \leq \frac{n^2 + n^2}{n^5} = \frac{2}{n^3}.
\]

The series \(\sum 2/n^3\) converges because \(\sum 1/n^3\) does, by Theorem 2.13. Hence, by the Comparison Test, the given series converges also.

The following, more sophisticated, version of the Comparison Test, is more useful. We could call it the ‘Limiting’ Comparison Test, but we’ll usually just call it the Comparison Test (since the previous two versions of the Test can be thought of as special cases of this one.)

**Theorem 2.17 (Comparison Test)** Suppose that \((a_n), (b_n)\) are positive and that \(a_n/b_n \to L\), where \(L \neq 0\) (and \(L\) is finite) as \(n \to \infty\). Then \(\sum a_n\) and \(\sum b_n\) either both converge or both diverge: that is, they have the same behaviour with respect to convergence.
2.6. Some useful tests for non-negative series

Proof
Note that $L$ will be positive because $a_n, b_n \geq 0$ and $L \neq 0$. Because $a_n/b_n \to L$, there will be some $N$ so that for all $n \geq N$,

$$\left| \frac{a_n}{b_N} - L \right| < \frac{L}{2}. $$

This is just taking $\epsilon = L/2 > 0$ in the definition of the limit of a sequence. So, for all $n \geq N$,

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}. $$

If $\sum b_n$ converges, then so does $\sum (3L/2)b_n$ and hence, by the fact that $a_n \leq (3L/2)b_n$ for all $n \geq N$, Theorem 2.16 shows that $\sum a_n$ converges also. On the other hand, if $\sum a_n$ converges, then so too does $\sum (2/L)a_n$ and, since $b_n \leq (2/L)a_n$ for $n \geq N$, Theorem 2.16 shows that $\sum b_n$ converges too. So, $\sum a_n$ converges if and only if $\sum b_n$ converges. In other words, either they both converge or they both diverge. 

Example 2.3 Consider again

$$\sum \frac{n^2 + 1}{n^5 + n + 1}. $$

Using the limiting form of the Comparison Test to compare the series with $\sum 1/n^3$, we simply observe that, since

$$\frac{(n^2 + 1)/(n^5 + n + 1)}{1/n^3} = \frac{n^5 + n^3}{n^5 + n + 1} = \frac{1 + n^{-2}}{1 + n^{-4} + n^{-5}} \to 1 \neq 0, $$

and since $\sum 1/n^3$ converges, then the given series converges too.

2.6.2 Ratio test

Another very useful test is the Ratio Test.

Theorem 2.18 (Ratio Test) Let $\sum a_n$ be a non-negative series such that

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \quad (L = \infty \text{ allowed}). $$

Then

1. $L < 1 \implies \sum a_n$ converges.
2. $L > 1 \implies \sum a_n$ diverges (This includes the case $L = \infty$.)

Proof
We prove the first part. (The second part can be proved similarly: try it!) Suppose that $L < 1$. Evidently, we may choose an $M$ such that $L < M < 1$. Hence there exists $N$ such that

$$n \geq N \implies \frac{a_{n+1}}{a_n} < M. $$
In particular, $a_{N+1} < Ma_N$. From this we see that in general we have

$$a_{N+n} < M^n a_N.$$\qed

Now since the geometric series $\sum M^n a_N$ converges (since $0 < M < 1$), we have by the Comparison Test that $\sum a_{N+n}$ converges, and hence $\sum a_n$ converges.

---

**Warning!**

Note that the Ratio Test says *nothing* if $L = 1$: in this case, the test is useless. In fact, consider the series $\sum 1/n$ and $\sum 1/n^2$. In both cases, $a_{n+1}/a_n \to 1$, yet the first series is divergent and the second convergent. So the ratio test fails in the case $L = 1$ not because we can’t prove that it works, but because the limit of the ratio really tells us nothing at all about convergence or divergence if that limit is 1.

---

**Example 2.4** Consider $\sum n^7/6^n$. Letting $a_n = n^7/6^n$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^7/6^{n+1}}{n^7/6^n} = \frac{1}{6} \left(1 + \frac{1}{n}\right)^7 \to \frac{1}{6}.$$ \hspace{1cm}

This limit is less than 1, so the series converges.

---

### 2.6.3 Root test

Also useful is the Root Test.

**Theorem 2.19 (Root Test)** Let $\sum a_n$ be a non-negative series, and suppose that $a_n^{1/n} \to L$ as $n \to \infty$ (where we allow $L = \infty$). Then,

1. $L < 1 \Rightarrow \sum a_n$ converges.
2. $L > 1 \Rightarrow \sum a_n$ diverges (this includes the case $L = \infty$).

**Example 2.5** Consider again $\sum n^7/6^n$. Here,

$$a_n^{1/n} = \left(\frac{n^7}{6^n}\right)^{1/n} = \frac{n^{7/n}}{6} = \frac{(n^{1/n})^7}{6}.$$ \hspace{1cm}

Now, $n^{1/n} \to 1$, so $a_n^{1/n} \to 1/6$ as $n \to \infty$. By the Root Test, the series converges. Again, note that the Root Test says nothing about the case $L = 1$.

---

### 2.6.4 Integral test

The following test draws on the interpretation of an area.
Theorem 2.20 (Integral Test)  Let \( g \) be a positive, decreasing, integrable (for example, continuous) function on \([1, \infty)\), and let \( G(n) = \int_1^n g(x) \, dx \). Then the series \( \sum g(n) \) converges if and only if the sequence \( (G(n)) \) converges. In other words, \( \sum g(n) \) converges if and only if the improper integral \( \int_1^\infty g(x) \, dx \) exists.

In fact, the following slight generalisation is valid.

Theorem 2.21 (Integral Test)  Suppose that \( a \geq 1 \) is a fixed number. Let \( g \) be a positive, decreasing, function defined on \([a, \infty)\) and integrable on \([a, \infty)\), and let \( G(n) = \int_a^n g(x) \, dx \). Then the series \( \sum g(n) \) converges if and only if the sequence \( (G(n)) \) converges. In other words, \( \sum g(n) \) converges if and only if the improper integral \( \int_a^\infty g(x) \, dx \) exists.

(This second version is useful when the integral exhibits improper behaviour near 1, as in the following example.)

Example 2.6  Consider \( \sum 1/(n \log n) \). We know that \( \sum 1/n \) diverges and that \( \sum 1/n^2 \) converges. This series is 'between' these two. To see whether it converges, we can use the integral test. Let \( g(n) = 1/(n \log n) \). Then, taking \( a = 2 \) in the general version of the integral test, we have

\[
\int_2^n g(x) \, dx = \int_2^n \frac{1}{x \log x} \, dx = \int_{\log 2}^{\log n} \frac{1}{u} \, du,
\]

where we have made the substitution \( u = \log x \). So

\[
G(n) = \log n - \log \log 2.
\]

Since \( G(n) \to \infty \) as \( n \to \infty \), the series is divergent. (We use \( a = 2 \) rather than \( a = 1 \) because the integral of \( g(x) \) is not defined when \( x = 1 \).)

2.7 Alternating series

A series is alternating if its terms are alternately positive and negative. Such a series takes the form \( \pm \sum (-1)^{n+1} c_n \), where \( c_n \geq 0 \).

Theorem 2.22 (Leibniz Alternating Series Test (‘LAST’))  Suppose that \( \sum a_n = \sum (-1)^{n+1} c_n \) is an alternating series, where \( c_n \geq 0 \). Then, if \( (c_n) \) is a decreasing sequence and \( \lim_{n \to \infty} c_n = 0 \), the series \( \sum a_n \) converges.

Corollary 2.23  \( \sum \frac{(-1)^{n+1}}{n^s} \) converges for \( s > 0 \).
Warning!

This test says that if the sequence \((c_n)\) is decreasing and tends to 0, then the series converges. It says nothing at all if one of these two conditions fails to hold. This does not mean that these two conditions are necessary for convergence of the alternating series: it just means that the Leibniz test doesn’t work in those situations.

Example 2.7  Let us use the Leibniz Alternating Series Test to prove that
\[
\sum (-1)^n \frac{\sqrt{n}}{n+1}
\]
converges.

The series is alternating, and takes the form \(\sum (-1)^n c_n\) where \(c_n = \sqrt{n}/(n+1) \geq 0\). We have
\[
c_n = \frac{1/\sqrt{n}}{1+1/n} \to 0.
\]

Also, \((c_n)\) is decreasing. There is more than one way to show this. First, we could note that
\[
\frac{c_{n+1}}{c_n} = \frac{\sqrt{n+1}/(n+2)}{\sqrt{n}/(n+1)} = \frac{(n+1)\sqrt{n+1}}{\sqrt{n}(n+2)} = \frac{\sqrt{(n+1)^2(n+1)}}{n(n+2)^2} = \frac{\sqrt{n^3+3n^2+3n+1}}{n^3+4n^2+4n},
\]
and this is at most 1 because \(4n^2 + 4n \geq 3n^2 + 3n + 1\). Alternatively, we could note that if \(f(x) = \sqrt{x}/(x+1)\), then
\[
f'(x) = \frac{1/(2\sqrt{x})(x+1) - \sqrt{x}}{(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} \leq 0 \text{ for } x \geq 1,
\]
and this shows that \(f\) is decreasing for \(x \geq 1\) and hence that \((c_n)\) is decreasing.

2.8 Absolute convergence

2.8.1 Definition of absolute convergence

Definition 2.7  Let \(\sum a_n\) be a series (in which some of the terms may be negative). If \(\sum |a_n|\) converges, we say that \(\sum a_n\) converges absolutely. If \(\sum a_n\) converges but \(\sum |a_n|\) diverges, then \(\sum a_n\) is said to converge conditionally.

Absolute convergence implies convergence.
**Theorem 2.24**  If a series is absolutely convergent, then it is convergent.

**Proof**
Suppose that the series $\sum a_n$ converges absolutely. This means that $\sum |a_n|$ is a convergent series of non-negative terms. From the fact that $-|a_n| \leq a_n \leq |a_n|$, we deduce that $a_n + |a_n| \geq -|a_n| + |a_n| = 0$ and $a_n + |a_n| \leq |a_n| + |a_n| = 2|a_n|$. It follows that $0 \leq (a_n + |a_n|)/2 \leq |a_n|$. In a similar way, it can be seen that $0 \leq (|a_n| - a_n)/2 \leq |a_n|$. So the series

$$\sum a_n + |a_n|, \quad \sum |a_n| - a_n$$

are non-negative series which, by the comparison test (comparing with the convergent series $\sum |a_n|$) are both convergent. The fact that $\sum a_n$ converges now follows from

$$\sum a_n = \sum \left( \frac{a_n + |a_n|}{2} - \frac{|a_n| - a_n}{2} \right),$$

and the fact that if $\sum c_n$ and $\sum d_n$ converge, then so also does $\sum (c_n - d_n)$. (see Theorem 2.14 for this last observation.)

- Note that if $\sum a_n$ is a convergent series with non-negative terms, then it is absolutely convergent.
- From what we saw earlier in Theorem 2.11, the geometric series $\sum a r^{n-1}$ converges absolutely if $|r| < 1$.
- By Theorem 2.23 and the fact that $\sum 1/n$ diverges, the series $\sum (-1)^n/n$ converges conditionally. (Theorem 2.23 shows it converges, but it does not converge absolutely because $\sum 1/n$ diverges.)

### 2.8.2 Tests for absolute convergence

The Comparison, Ratio, and Root Tests can be generalised as follows.

**Theorem 2.25 (Comparison Test)**  Let $(a_n), (b_n)$ be sequences such that $|a_n| \leq |b_n|$ for all $n$. Then

1. If $\sum b_n$ converges absolutely, then $\sum a_n$ does also and $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |b_n|$.
2. If $\sum |a_n|$ diverges, then $\sum |b_n|$ diverges.

**Theorem 2.26 (Ratio Test)**  Let $\sum a_n$ be a series such that

$$L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \quad (L = \infty \text{ allowed}).$$

Then

1. $L < 1 \Rightarrow \sum a_n$ converges absolutely
2. $L > 1 \Rightarrow \sum a_n$ diverges.
2. Series of real numbers

Note the slightly stronger conclusion than you might have expected in the case where \( L > 1 \). You might think that this only establishes that \( \sum |a_n| \) diverges, but in fact \( \sum a_n \) diverges.

**Theorem 2.27 (Root Test)** Let \( \sum a_n \) be a series, and suppose that \( |a_n|^{1/n} \to L \) as \( n \to \infty \) (where we allow \( L = \infty \)). Then,

1. \( L < 1 \Rightarrow \sum a_n \) converges absolutely.
2. \( L > 1 \Rightarrow \sum a_n \) diverges (this includes the case \( L = \infty \)).

2.9 Power series

For our purposes, a *power series* is a series of the form \( \sum a_n x^n \), where \( x \) is a real variable. Perhaps the most important example of a power series is \( \sum x^n/n! \), used to define the exponential function. It turns out that, for any real number \( x \), this series converges, and we may *define* the exponential function by

\[
\exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}.
\]

There isn’t enough time to cover power series in very great detail, but we look at how our convergence tests apply to power series.

Let’s take the exponential series first. It’s easy to show that this converges absolutely for all \( x \). We simply observe that

\[
\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \frac{|x|}{n+1} \to 0,
\]

for any \( x \), and so, by the Ratio Test, absolute convergence follows.

Here’s a less straightforward example.

**Example 2.8** Let’s determine exactly those values of \( x \) for which the series \( \sum x^n/n \) is convergent. Taking \( a_n = x^n/n \), the ratio \( |a_{n+1}|/|a_n| \) is

\[
\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \frac{|x|}{n+1} \to |x|.
\]

The ratio test therefore tells us that the series converges absolutely if \( |x| < 1 \), and that it diverges if \( |x| > 1 \). But what if \( |x| = 1 \)? Here, the ratio test is useless and we have to be more sophisticated. Well, \( |x| = 1 \) corresponds to two cases: \( x = 1 \) and \( x = -1 \). We treat each separately. When \( x = 1 \), the series is the harmonic series \( \sum 1/n \), which we know diverges. When \( x = -1 \), we have the series \( \sum (-1)^n/n \). This is convergent, by the Leibniz Alternating Series Test. (You should check this!) So we have now determined exactly the values of \( x \) where the series converges: it converges for \(-1 \leq x < 1\) and diverges for all other values of \( x \).

A general result about power series is as follows.
2.9. Learning outcomes

**Theorem 2.28** For every sequence \((a_n)\), there is an \(R\) such that the series \(\sum a_n x^n\) converges absolutely for all \(x \in (-R, R)\), and diverges for all \(x\) with \(|x| > R\). (It is possible that \(R = \infty\)).

In the case in which \(R\) is finite, what happens at \(\pm R\) is not determined by this theorem, and has to be considered separately. The name *radius of convergence* is given to \(R\).

**Learning outcomes**

At the end of this chapter and the relevant reading, you should be able to:

- state what’s meant by a series
- state precisely the definition of convergence of a series
- prove that some sequences converge by considering their partial sums
- prove that if a series converges, then the \(n\)th term tends to 0; and use this to prove divergence
- describe under what conditions a geometric series converges or diverges, and be able to prove these results
- state what is meant by the harmonic series and be able to prove it diverges
- state that \(\sum 1/n^s\) converges for \(s > 1\) and prove that it diverges for \(s \leq 1\); and be able to use this result.
- state and use the ‘algebra of limits’ results for series
- state and use the Comparison Test (in all its forms), and be able to prove these results.
- state and use the Ratio Test (but no need to be able to prove it)
- state and use the Root Test (but no need to be able to prove it)
- state and use the Integral Test (but no need to be able to prove it)
- state and use the Leibniz Alternating Series Test (but no need to be able to prove it)
- state what is meant by absolute convergence, and know, and be able to use, the fact that this implies convergence (proof not necessary)
- state what is meant by conditional convergence
- determine divergence, conditional, and absolute convergence by using the tests already mentioned
- state what is meant by a power series; and be able to determine the set of \(x\) for which a given power series converges.

**Sample examination questions**

**Question 2.1**

Use the comparison test to prove that the series

\[
\sum \frac{n^2 - n + 1}{n^3 + 1}
\]

diverges.
2. Series of real numbers

**Question 2.2**
Show that, for all \( n \geq 1 \),
\[
\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq \frac{1}{2(n+1)\sqrt{n}}.
\]
Use this, and the comparison test, to show that \( \sum n^{-3/2} \) converges.

**Question 2.3**
Let
\[
s_n = 1 + 2x + 3x^2 + \cdots + nx^{n-1}.
\]
Evaluate \( s_n - xs_n \). Deduce that, for \(-1 < x < 1\), the series \( \sum n x^{n-1} \) converges, and that
\[
\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.
\]

**Question 2.4**
Suppose that \((a_n)\) is a non-negative sequence, and that \( \sum a_n \) is convergent. Show that, for any subsequence \((a_{nk})\), the sum \( \sum_{k=1}^{\infty} a_{nk} \) converges. Give an example of a sequence \((a_n)\) and a subsequence \((a_{nk})\) such that \( \sum a_n \) converges but \( \sum a_{nk} \) does not.

**Question 2.5**
Prove that if \( \sum a_n \) converges and \( \sum b_n \) diverges, then \( \sum (a_n + b_n) \) diverges.
[Be careful to ensure that your proof does not assume that \( a_n \) and \( b_n \) are non-negative.]

**Question 2.6**
Show that
\[
\sum \left( \frac{a}{n} + \frac{b}{n+1} - \frac{c}{n+2} \right)
\]
converges if and only if \( c = a + b \).

**Question 2.7**
For each of the following series say whether the series converges or diverges. In each case, give a brief reason or proof.
- \( \sum n^{1/n}, \sum \frac{1}{n^{5/4}}, \sum \cos \left( \frac{1}{n} \right) \), 
- \( \sum \frac{1}{\sqrt{n}}, \sum \frac{\sqrt{n}}{2n^3 - 1}, \sum \frac{2n + (-1)^n}{n^2 - n + 1}, \sum \frac{n + (-1)^n \sqrt{n}}{(n+1)^4} \).

**Question 2.8**
Use the root test to determine whether the series \( \sum \left( \frac{n}{2n+1} \right)^n \) converges.
Question 2.9
For each of the following series say whether the series converges or diverges. In each case, give a brief reason or proof.

\[ \sum \frac{(n+1)^2}{n!}, \sum \frac{2.5.8\ldots(3n-1)}{4.8.12\ldots(4n)}, \sum \frac{(n!)^2}{(2n)!} 6^n, \sum \frac{(n!)^2}{(2n)!} 4^n. \]

Question 2.10
Prove Theorem 2.19; i.e., verify the correctness of the Root Test. (Hint: try to follow the proof for the Ratio Test).

Question 2.11
Discuss the convergence of the series \( \sum \frac{1}{(n+1)(\log(n+1))^s} \) for all \( s > 0 \).

Question 2.12
Determine whether each of the following series converges, in each case justifying your answer carefully.

\[ \sum 3^n 5^{-n^2}, \sum \frac{(-1)^n n}{n^2 + 1}, \sum \frac{(-1)^n n^2}{n^2 + 1}, \sum (-1)^n \sin \left( \frac{1}{n} \right), \sum \frac{\sin n}{n^{3/2}}. \]

Question 2.13
Determine whether each of the following series converges:

\[ \sum \frac{(-1)^n}{\log(n+1)}, \sum \frac{(-1)^n \sqrt{n}}{\sqrt{n + 1}}. \]

Question 2.14
For which values of \( x \) does the series \( \sum \frac{n^2 + 2}{4^n n} x^n \) converge?

Question 2.15
Decide whether the series \( \sum a_n \) is absolutely convergent, conditionally convergent or divergent when \( a_n \) has the following forms:

\[ \frac{\sin n}{n^2}, \frac{(-1)^n n^4}{n^4 + 1}, \frac{(-1)^n n^3}{n^4 + 1}, \frac{(-1)^n n^2}{n^4 + 1}, \frac{(-1)^n \sin \sqrt{n} \pi}{n^2}. \]

Question 2.16
Determine for which values of \( x \) the series \( \sum \frac{(n + 1)^2 x^n}{n^3} \) converges.
2. Series of real numbers

Comments on selected activities

Feedback to activity 2.1
We have
\[ s_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + k. \]
This, you might recognise, is the sum of an arithmetic progression, and so, using the formula for such a sum, we obtain
\[ s_n = \frac{1}{2} n(n + 1). \]

Feedback to activity 2.2
There are at least two ways we can do this. First, as we already noted, the partial sum \( s_n \) equals \(-1\) if \( n \) is odd and 0 of \( n \) is even. So the sequence \( (s_n) \) alternates between the two values \(-1\) and 0, and for this reason it does not converge. So the series diverges. Alternatively, we could use Theorem 2.9: \( a_n = (-1)^n \) and so \( a_n \) does not tend to 0. Hence the series diverges.

Feedback to activity 2.3
Let \( s_n \) be the \( n \)th partial sum of \( \sum 1/n^s \). Because \( s \leq 1 \), \( 1/n^s \geq 1/n \). So, if \( t_n \) is the \( n \)th partial sum of \( \sum 1/n \), then \( s_n \geq t_n \). But \( \sum 1/n \) diverges and \( 1/n \geq 0 \) for all \( n \), so by Theorem 2.10, \( t_n \to \infty \). Since \( s_n \geq t_n \), it follows that \( s_n \to \infty \) also and hence the sequence \( (s_n) \) diverges. But this means precisely that \( \sum 1/n^s \) diverges.

Sketch answers to or comments on selected questions

Answer to question 2.1
The series \( \sum (n^2 - n + 1)/(n^3 + 1) \) has \( n \)th term \( a_n \) which ‘behaves like’ \( 1/n \), and so we are confident that the series will diverge. But we need to be more precise and formal. We formally compare it with \( \sum 1/n \) using the Comparison Test. Here’s how we would use the basic form of the CT (noting that the \( n \)th term is non-negative). Note that we need to show \( a_n \) is greater than or equal to some constant times \( 1/n \). Proving less than or equal to would be no use at all in proving divergence. For \( n \geq 2 \), \( n^2 - n + 1 \geq \frac{n^2 - (n^2/2) + 0}{n^3 + n^3} = \frac{1}{4n} \).

Therefore, the given series diverges, since \( \sum 1/n \) does.

Alternatively, we could use the ‘limiting form’ of the CT, as follows: since
\[ \frac{a_n}{1/n} = \frac{n^3 - n^2 + n}{n^3 + 1} = \frac{1 - 1/n + 1/n^2}{1 + 1/n^3} \to 1 \neq 0 \]

and since \( \sum 1/n \) diverges, the given series also diverges.
2.9. Sketch answers to or comments on selected questions

Answer to question 2.2

(This is a rather strange, but ‘elementary’, way of showing that \( \sum n^{-3/2} \) converges.)

Multiplying by \( \sqrt{n} \) and rearranging makes the given inequality equivalent to:

\[
1 - \frac{1}{2(n + 1)} \geq \sqrt{n/n + 1},
\]

which is seen to be true on squaring both sides.

Now notice that the series \( \sum \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n + 1}} \right) \) sums to 1, since the \( n \)th partial sum is equal to \( 1 - 1/\sqrt{n + 1} \). We can now apply the Comparison Test to see that \( \sum 1/2(n + 1)\sqrt{n} \) converges. Since \( (1/2(n + 1)\sqrt{n})/n^{-3/2} \to 1/2 \), the limiting form of the Comparison Test now tells us that \( \sum n^{-3/2} \) converges too.

Answer to question 2.3

We have

\[
s_n - xs_n = \left(1 + 2x + 3x^2 + \cdots + nx^{n-1}\right) - \\
\quad \left(x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1} + nx^n\right)
\]

\[= 1 + x + x^2 + \cdots + x^{n-1} - nx^n\]

\[= \frac{1 - x^n}{1 - x} - nx^n\]

\[\to \frac{1}{1 - x},\]

as \( n \to \infty \), for \(-1 < x < 1\). That is,

\[
s_n - xs_n = (1 - x)s_n \to \frac{1}{1 - x},
\]

so

\[s_n \to \frac{1}{(1 - x)^2},\]

as required. Note that \( s_n \) is the \( n \)th partial sum of the given series, so this is exactly what we needed to prove.

Answer to question 2.4

If the sequence \((a_n)\) is non-negative, then the sequence \((s_n)\) of partial sums is increasing. Since the sequence converges to some limit \( L \), then \( L \) is an upper bound for the set of partial sums. For any subsequence \((a_{n_k})\), the sequence of partial sums is again increasing, and also bounded above by \( L \), so the series \( \sum a_{n_k} \) also converges – though probably not to \( L \).

For the example, one possibility is to take the sequence \((a_n)\) to be given as follows:

\[1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \ldots\]

The series \( \sum a_n \) can be seen directly to be convergent, although the series consisting only of the positive terms is divergent (because it is the harmonic series \( \sum 1/n \)).
Answer to question 2.5

The argument is going to be by contradiction. We’ll suppose that \( \sum a_n \) converges, that \( \sum b_n \) diverges, and (for a contradiction) that \( \sum (a_n + b_n) \) converges. The way we’ll get to a contradiction is that our assumptions about \( \sum a_n \) and \( \sum (a_n + b_n) \) will imply that \( \sum b_n \) converges after all. This is the right approach because knowing that something *converges* tells us something concrete, so it’s easier to argue about convergence than about divergence.

In fact, let’s prove a useful, and obvious, general result. Suppose that \( \sum c_n \) and \( \sum d_n \) are two convergent series, and that \( \alpha, \beta \in \mathbb{R} \). Then the partial sums of the series—let’s call them \( s_n \) and \( t_n \), respectively—converge, to limits \( L \) and \( M \) say. The partial sums of \( \sum (\alpha c_n + \beta d_n) \) are \( \alpha s_n + \beta t_n \), so they converge to \( \alpha L + \beta M \). In other words, the series \( \sum (\alpha c_n + \beta d_n) \) converges. Now suppose that \( \sum a_n \) converges, \( \sum b_n \) diverges, and (for a contradiction) \( \sum (a_n + b_n) \) converges. Then, by what we’ve just observed (taking \( c_n = a_n + b_n, d_n = a_n, \alpha = 1, \beta = -1 \)) \( \sum ((a_n + b_n) - a_n) \) converges: that is, \( \sum b_n \) converges, which is the contradiction we were aiming for. [This argument can be carried out without stating the general result we did, but ‘proof by contradiction’ is more-or-less essential.]

I asked you to be careful that you did not assume the series to be non-negative. Many students attempt this exercise as follows: let \( s_n \) be the partial sums of \( \sum a_n \) and \( t_n \) the partial sums of \( \sum b_n \). Then, given that \( \sum a_n \) converges and \( \sum b_n \) diverges, we have \( s_n \to L \) for some finite \( L \) and \( t_n \to \infty \). Then, the partial sums of \( \sum (a_n + b_n) \) are \( s_n + t_n \), and \( s_n + t_n \to L + \infty = \infty \), so \( \sum (a_n + b_n) \) diverges. But this argument needs to assume that the series \( \sum b_n \) is non-negative. For, it makes the claim that since \( \sum b_n \) diverges, its partial sums tend to infinity. That’s true if \( b_n \geq 0 \), but not in general. (For example, \( \sum (-1)^n \) diverges but the partial sums do not tend to infinity.)

Answer to question 2.6

The series is \( \sum x_n \), where

\[
x_n = \frac{a}{n} + \frac{b}{n+1} - \frac{c}{n+2}
\]

\[
= \frac{a(n+1)(n+2) + bn(n+2) - cn(n+1)}{n(n+1)(n+2)}
\]

\[
= \frac{an^2 + 3an + 2a + bn^2 + 2bn - cn^2 - cn}{n^3 + 3n^2 + 2n}
\]

\[
= \frac{(a + b - c)n^2 + (3a + 2b - c)n + 2a}{n^3 + 3n^2 + 2n}.
\]

Now we can use the Comparison Test. If \( (a + b - c) \neq 0 \) then

\[
\lim_{n \to \infty} \frac{x_n}{1/n} = a + b - c \neq 0,
\]

so, since \( \sum 1/n \) diverges, so does \( x_n \). If, however, \( a + b - c = 0 \) then \( c = a + b \) and

\[
x_n = \frac{(2a + b)n + 2a}{n^3 + 3n^2 + 2n}.
\]
If \(2a + b \neq 0\), then \(x_n/(1/n^2) \to 2a + bn \neq 0\), so by comparison with the convergent series \(\sum 1/n^2\), the series converges. If \(2a + b = 0\) and \(2a \neq 0\), then \(x_n/(1/n^3) \to 2a \neq 0\), which again shows convergence since \(\sum 1/n^3\) converges. If \(2a + b = 0\) and \(a = 0\) then \(a = b = c = 0\) and the series has every term equal to 0, and so is certainly convergent.

But wait a minute! You might be uneasy about the argument just given because the Comparison Tests only apply to non-negative series, yet we do not know for sure that the terms \(x_n\) of this series are non-negative. However, it’s certainly true that, either, for large enough \(n\) (i.e., for all \(n \geq N\), for some \(N\), \(x_n \geq 0\) or, for large enough \(n\), \(x_n \leq 0\). (And, clearly, a counterpart of the Comparison Test holds for non-positive series.) If \(a + b - c \neq 0\) then the terms eventually have the same sign (that is, non-negative or non-positive) as \(a + b - c\); and if \(a + b - c = 0\), but \(2a + b \neq 0\), then the ultimate sign of the terms is determined by \(2a + b\). Otherwise, the sign of \(x_n\) is the sign of \(2a\). So, from some point on, all the terms have the same sign, which we can, without any loss of generality, assume to be non-negative. A finite number of initial negative terms do not cause a problem in applying the Comparison Test, because the convergence behaviour is not affected by them.

There are other approaches to proving convergence in the case \(c = a + b\). For example, notice that the partial sum \(s_n\) can be simplified as follows:

\[
s_n = \sum_{k=1}^{n} \frac{a}{k} + \sum_{k=1}^{n} \frac{b}{k+1} - \sum_{k=1}^{n} \frac{a+b}{k+2}
\]

\[
= \sum_{k=1}^{n} \frac{a}{k} + \sum_{i=2}^{n+1} \frac{b}{i} - \sum_{j=3}^{n+2} \frac{a+b}{j}
\]

\[
= \sum_{k=3}^{n} \left( \frac{a}{k} + \frac{b}{k} - \frac{(a+b)}{k} \right) + a + \frac{a+b}{2} + \frac{b}{n+1} - \frac{a+b}{n+1} - \frac{a+b}{n+2}
\]

\[
= 0 + \frac{3a}{2} + \frac{b}{2} + \frac{b}{n+1} - \frac{a+b}{n+1} - \frac{a+b}{n+2}
\]

\[
\to \frac{3a}{2} + \frac{b}{2}.
\]

This approach has the advantage that it shows not just that the series is convergent (that is, that \(s_n\) converges), but it also shows what the sum of the series is.

**Answer to question 2.7**

\(n^{1/n} \to 1\) as \(n \to \infty\). So here, \(a_n\) does not tend to 0 and so the series \(\sum n^{1/n}\) diverges.

\(\sum 1/n^{3/2}\) converges because \(3/2 > 1\). (Here, we are using the standard result on series of the form \(\sum 1/n^s\). You may use this result without proof, unless you are asked explicitly to prove it.)

\(\cos(1/n) \to \cos(0) = 1\) as \(n \to \infty\), so the \(n^{th}\) term does not tend to 0, and the series diverges.

\(\sum 1/\sqrt{n}\) is \(\sum 1/n^{1/2}\). Since \(1/2 \leq 1\), this series diverges (by the standard result already mentioned).

For the next two, we use the limiting form of the comparison test.
Comparing $\sum \sqrt{n}/(2n^3 - 1)$ with $\sum 1/n^{5/2}$: we have
\[
\lim_{n \to \infty} \frac{\sqrt{n}/(2n^3 - 1)}{1/n^{5/2}} = \lim_{n \to \infty} \frac{1}{2 - n^{-3}} = \frac{1}{2} \neq 0.
\]
Since $\sum 1/n^{5/2}$ converges, the given series converges too.

We compare $\sum (2n + (-1)^n)/(n^2 - n + 1)$ with $\sum 1/n$. Note that, despite the $(-1)^n$, all terms are non-negative. We have
\[
\lim_{n \to \infty} \frac{(2n + (-1)^n)/(n^2 - n + 1)}{1/n} = \lim_{n \to \infty} \frac{2 + (-1)^n}{1 - n^{-1} + n^{-2}} = 2 \neq 0,
\]
and since $\sum 1/n$ diverges, the given series does too. Recall that the test only applies if the limit we are working out exists, is finite, and is non-zero.

For the last one, let’s see how we can use the comparison test in the original form.
Roughly speaking, $a_n = (n + (-1)^n)\sqrt{n}/(n + 1)^4$ behaves like $n/n^4$, which is $1/n^3$. We know $\sum 1/n^3$ converges, so our given sequence will too. Note that because $n \geq \sqrt{n}$ for all $n$, $a_n \geq 0$ for all $n$. We want a bound of the form $a_n \leq \text{constant} \times (1/n^3)$. (For proving convergence, a bound $a_n \geq \ldots$ would be useless.) We have:
\[
\frac{n + (-1)^n \sqrt{n}}{(n + 1)^4} \leq \frac{n + n}{n^4} = \frac{2}{n^3},
\]
so we see that the series converges.

**Answer to question 2.8**

We have $\left(\frac{n}{2n + 1}\right)^{1/n} = \frac{n}{2n + 1} \to \frac{1}{2} < 1$, so the series converges, by the Root Test.

**Answer to question 2.9**

For all of these, we use the ratio test. For the first, we have
\[
a_{n+1} = \frac{(n+1)!}{(n+1)^2} \times \frac{n!}{(n+1)!} = \frac{(n+2)^2}{(n+1)^2} \to \frac{1}{n+1} \to 0,
\]
(where we have used $(n+1)! = (n+1)n!$). As the ratios tend to a limit less than 1, the series converges.

The next one is trickier: here, $a_n = \frac{2.5.8\ldots(3n-1)}{4.8.12\ldots(4n)}$, so
\[
a_{n+1} = \frac{2.5.8\ldots(3n+1)-1}{4.8.12\ldots(4n+1))} = \frac{2.5.8\ldots(3n-1)(3n+2)}{2.5.8\ldots(4n)(4n+4)} = \frac{3n+2}{4n+4}a_n.
\]
So,
\[
a_{n+1} = \frac{3n+2}{4n+4} \to \frac{3}{4} < 1,
\]
and the series converges, by the Ratio Test.
Next,

\[
\frac{(n+1)!^26^{n+1}/(2(n+1))!}{(n!)^26^n/(2n)!} = 6 \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \\
= 6(n+1)^2 \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \\
= 6 \frac{(n+1)^2}{(2n+2)(2n+1)} \\
= 6 \frac{(1+1/n)(1+1/n)}{(2+2/n)(2+1/n)} \\
\rightarrow 6 \frac{1}{4} = \frac{3}{2} > 1,
\]

so the series diverges, by the Ratio Test.

The next one is similar. Proceeding as above, we will find that

\[
a_{n+1} \frac{a_n}{a_n} = 4 \left(\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\right) \rightarrow 1.
\]

Now, the Ratio Test tells us nothing when this limit is 1. So we can’t determine whether the series converges or diverges by using that test. But, look at the ratio: it can be written as

\[
a_{n+1} \frac{a_n}{a_n} = \frac{(n+1)(n+1)}{(n+1)(n+1/2)} = \frac{n+1}{n+(1/2)}.
\]

This is clearly greater than 1, so we see that for all \( n \), \( a_{n+1} > a_n \). Because of this, we can’t have \( a_n \rightarrow 0 \), so the series diverges. (Note: we can’t say ‘it diverges, by the ratio test’: we have simply used the fact that if \( a_n \not\rightarrow 0 \) then \( \sum a_n \) diverges.)

**Answer to question 2.10**

Let \( (a_n) \) be a non-negative sequence, and suppose that \( a_{\frac{1}{n}} \) converges to the limit \( L \). We need to prove that, if \( L > 1 \), then \( \sum a_n \) diverges, whereas if \( L < 1 \), then \( \sum a_n \) converges.

Suppose first that \( L > 1 \). Then there exists \( N \) such that, for \( n \geq N \), \( a_{\frac{1}{n}} > 1 \), or equivalently \( a_n > 1 \). Therefore \( a_n \) does not tend to zero, and the series does not converge.

Now suppose that \( L < 1 \). Choose some \( M \) such that \( L < M < 1 \). Since \( a_{\frac{1}{n}} \rightarrow L \), there is some \( N \) such that, for \( n \geq N \), \( a_{\frac{1}{n}} < M \), i.e., \( a_n < M^n \). Compare with the sequence \( b_n = M^n \): we know that \( 0 \leq a_n < b_n \) for \( n \geq N \), and that \( \sum b_n \) converges – since \( M < 1 \). Therefore, by the comparison test, \( \sum a_n \) converges.

**Answer to question 2.11**

Let

\[
g(x) = \frac{1}{(x+1)(\log(x+1))^s},
\]

so that the series is \( \sum g(n) \). Note that \( g(x) \geq 0 \) and \( g \) is a decreasing function of \( x \), so we may use the Integral Test. Let \( G(n) = \int_1^n g(x) \, dx \). On making the substitution
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\[ u = \log(x + 1), \]

\[ G(n) = \int_{1}^{n} g(x) \, dx = \int_{\log 2}^{\log(n+1)} \frac{1}{u^s} \, du = \int_{\log 2}^{\log(n+1)} u^{-s} \, du, \]

which, if \( s \neq 1 \), is

\[ \frac{1}{-s + 1} \left( (\log(n + 1))^{-s+1} - (\log 2)^{-s+1} \right). \]

It is easy to see that this converges (and hence the series does, by the integral test) if and only if \( s > 1 \). When \( s = -1 \), the integral is

\[ \log \log(n + 1) - \log \log 2, \]

which does not converge as \( n \to \infty \). In this case, the series therefore diverges.

Therefore, \( \sum 1/((n + 1)(\log(n + 1))^s) \) converges if and only if \( s > 1 \).

**Answer to question 2.12**

For the first one, we can use the Ratio Test or the Root Test. The ratio \( a_{n+1}/a_n \) turns out to be \( 3(5^{-2n+1}) \) which tends to 0. Since this limit is less than 1, we have convergence. For the root test, we note that \( a_n^{1/n} = 3(5^{-n}) \to 0 \), which shows convergence.

Next, we have an alternating series \( \sum (-1)^n c_n \) where \( c_n = n/(n^2 + 1) \). Here, \( c_n \geq 0 \) and

\[ c_n = \frac{1}{n + (1/n)} \to 0. \]

Additionally, we can see that \( (c_n) \) is a decreasing sequence. For, the derivative of \( f(x) = x/(x^2 + 1) \) is \( f'(x) = (1 - x^2)/(x^2 + 1)^2 \), and \( f'(x) \leq 0 \) for \( x \geq 1 \).

For the third series, we note that \( (-1)^n n^2/(n^2 + 1) \) does not tend to 0 as \( n \to \infty \). We know that if a sequence \( \sum a_n \) converges, then \( a_n \to 0 \). It follows that the given series does not converge. Be very careful about how you reason here. Leibniz test does not apply, because \( c_n \) does not tend to 0. But we are not using Leibniz test to prove divergence: it can only ever be used to prove convergence. We prove divergence in this case by using the simple result that if the \( n \)th term does not tend to 0 then the series is divergent. That’s all. You would be totally wrong to say something like ‘by the Leibniz test, the series diverges’.

For the fourth series, we note that \( \sin(1/n) \to 0 \) and that it decreases with \( n \) (because \( \sin x \) is an increasing function in the range \([0, \pi/2] \), and if \( n \in \mathbb{N} \), then \( 1/n \) lies in this interval). So, by the Leibniz test, the series converge.

Finally, we have the series \( \sum a_n \) where \( a_n = \sin n/n^{3/2} \). Now, this is not a non-negative series, and it is not an alternating series either, so there aren’t many tests we can use. In fact, as is often the case, it is easier to prove absolute convergence (from which convergence follows). We have

\[ 0 \leq |a_n| = \left| \frac{\sin n}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}, \]

and, since \( \sum 1/n^{3/2} \) converges (because \( 3/2 > 1 \)), we have, by the Comparison Test, that \( \sum |a_n| \) converges. Thus the series is absolutely convergent and hence convergent.
2.9. Sketch answers to or comments on selected questions

Answer to question 2.13
For the first one, we can use LAST. Here, \( c_n = 1/\log(n + 1) \geq 0 \). Since \( \log x \) is an increasing function, \( (c_n) \) is a decreasing sequence and, since \( \log x \to \infty \) as \( x \to \infty \), \( c_n \to 0 \) as \( n \to \infty \). By LAST, the series converges.

For the second series, since
\[
\frac{\sqrt{n}}{\sqrt{n + 1}} = \frac{1}{\sqrt{1 + (1/n)}} \to 1,
\]
the \( n \)th term \( (-1)^n \sqrt{n}/\sqrt{n + 1} \) does not tend to 0 and, for this reason (and not by LAST), the series diverges.

Answer to question 2.14
We use the Ratio Test for absolute convergence. Let \( a_n = \frac{(n^2 + 2)x^n}{4^nn} \). Then
\[
\frac{|a_{n+1}|}{|a_n|} = \frac{((n + 1)^2 + 2)|x|^{n+1}/4^{n+1}(n + 1)}{(n^2 + 2)|x^n/4^n n} = \frac{1}{4} \frac{|x|^{n+1}2n + 3}{n^2 + 2} \left( \frac{n}{n + 1} \right) \to \frac{|x|}{4},
\]
So the series converges absolutely for \( |x| < 4 \) and it diverges for \( |x| > 4 \). When \( x \) equals 4 or \(-4\), the \( n \)th term does not tend to 0, so it diverges. Hence the series converges only for \( x \in (-4, 4) \).

Answer to question 2.15
\[
\left| \frac{\sin n}{n^2} \right| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}, \text{ so by the Comparison Test, the series is absolutely convergent.}
\]
\[
\sum \frac{(-1)^n n^4}{n^4 + 1} \text{ is divergent because } a_n \text{ does not tend to 0 as } n \to \infty.
\]
\[
\sum \frac{(-1)^n n^3}{n^4 + 1} \text{ is not absolutely convergent. To see this, compare with } \sum \frac{1}{n}, \text{ noting that}
\]
\[
\frac{|(-1)^n n^3/(n^4 + 1)|}{1/n} \to 1 \neq 0,
\]
and \( \sum 1/n \) diverges. To check whether the series converges conditionally, we might try the Leibniz test, since it is an alternating series. It has the form \( \sum (-1)^n c_n \), where \( c_n = n^3/(n^4 + 1) \geq 0 \). It is easily seen that \( c_n \to 0 \) as \( n \to \infty \). Also, \( (c_n) \) is a decreasing sequence, because \( a_2 \leq a_1 \) and the derivative of \( x^3/(x^4 + 1) \) is \( x^2(3 - x^4)/(1 + x^4)^2 \), which is negative for \( x \geq 2 \). (In fact, all that we really need is that the sequence \( (c_n) \) be decreasing from some point onwards, so there is no need to worry about whether \( a_2 \leq a_1 \) or not.) By Leibniz, the series converges. So the series converges conditionally.
\[
\left| \frac{(-1)^n n^2}{n^4 + 1} \right| = \frac{n^2}{n^4 + 1} \leq \frac{n^2}{n^4} = \frac{1}{n^2}, \text{ so, by comparison with } \sum 1/n^2, \text{ the series is absolutely convergent. There is no need to do any more work, because absolute convergence implies convergence. That is, there is no need to use the Leibniz test on the series.}
\]
2. Series of real numbers

\[ |(-1)^n \frac{\sin \sqrt{n}\pi}{n^2}| \leq \left| \frac{\sin \sqrt{n}\pi}{n^2} \right| \leq \frac{1}{n^2}, \text{ so, by comparison with } \sum 1/n^2, \text{ the series is absolutely convergent.} \]

**Answer to question 2.16**

Probably the easiest way to do this is to use the Ratio Test for absolute convergence. Let \( a_n = \frac{(n + 1)^2 x^n}{n^3} \). Then

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{(n + 2)^2 |x|^{n+1} / (n + 1)^3}{(n + 1)^2 |x|^n / n^3} = \frac{(n + 2)^2 n^3}{(n + 1)^5} |x| \to |x|.
\]

So, by the Ratio Test, the series converges absolutely for \( |x| < 1 \) and diverges for \( |x| > 1 \). When \( x = 1 \), the series is \( \sum (n + 1)^2 / n^3 \), which diverges by comparison with the series \( \sum 1/n \). When \( x = -1 \), the series is \( \sum (-1)^n (n + 1)^2 / n^3 \). Write this as \( \sum (-1)^n c_n \). Then \( c_n \to 0 \), and it is easy to see that the series is decreasing: for,

\[
c_n = \frac{(n + 1)^2}{n^3} = \frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3},
\]

and each of the terms on the right is decreasing. By the Leibniz test, the series converges. To sum up, the series converges absolutely for \( x \in (-1, 1) \), conditionally for \( x = -1 \), and it diverges for \( x \in (-\infty, -1) \cup [1, \infty) \).
Chapter 3
Sequences, functions and limits in higher dimensions

Reading


Some of this chapter contains material that is revision from 116 Abstract Mathematics. Coverage in the textbooks of the other material in this chapter is weak. Chapter 19 of Binmore’s book is probably the best place to look.

3.1 Introduction

In this chapter we look at what it means for a sequence of vectors to converge. We then look at limits and continuity of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, reminding ourselves en route of the relevant concepts for functions from $\mathbb{R}$ to $\mathbb{R}$ that we met in 116 Abstract Mathematics.

3.2 Sequences in $\mathbb{R}^m$

3.2.1 Distance in $\mathbb{R}^m$

The Euclidean distance (or simply distance) between $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ in $\mathbb{R}^m$ is defined to be

$$\|x - y\| = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}.$$ 

(The case in which $m = 1$ corresponds to the distance $|x - y|$ between two real numbers.)

There is a certain mathematical attraction in defining distances on rather more abstract or unusual spaces, and this leads to the notion of a metric space. This is something we will touch on later in this course. For the moment, we are primarily interested in Euclidean space $\mathbb{R}^m$, for some integer $m \geq 1$, and we shall always use the Euclidean distance.

Note: The Euclidean distance between two vectors $x, y$ in $\mathbb{R}^m$ is simply the norm or length of $x - y$, where the norm is the one arising from the usual inner product on $\mathbb{R}^m$. 

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(See Linear Algebra.) Consequently, the Euclidean distance has some nice properties. For example, by the triangle inequality for norms, we have that for any \( x, y, z \in \mathbb{R}^m \),
\[
\|y - z\| = \|(y - x) + (x - z)\| \leq \|y - x\| + \|x - z\|.
\]

Having equipped ourselves with a notion of distance in \( \mathbb{R}^m \), we can say what we mean by a bounded subset. A bounded subset of \( \mathbb{R}^m \) is one in which there is some fixed number bounding the distance between any two points in the set. Formally:

**Definition 3.1 (Bounded subset of \( \mathbb{R}^m \))** A subset \( B \) of \( \mathbb{R}^m \) is bounded if there is \( K > 0 \) such that for all \( x, y \in B \), \( \|x - y\| \leq K \).

Note that \( K \) is fixed: it does not depend on \( x, y \). (The definition would be meaningless if that were the case.) There are other, equivalent, ways to think about boundedness. For instance, we have the following characterisation.

**Theorem 3.1** A subset \( B \) of \( \mathbb{R}^m \) is bounded if and only if there is some \( M \) such that \( \|x\| \leq M \) for all \( x \in B \).

**Activity 3.1** Prove Theorem 3.1. (You will have to show that Definition 3.1 implies the property described in Theorem 3.1, and also that the property described in that theorem implies the property of Definition 3.1.)

### 3.2.2 Convergence in \( \mathbb{R}^m \)

A sequence \((x_n)\) in \( \mathbb{R}^m \) is simply an ordered list \( x_1, x_2, \ldots \) of elements of \( \mathbb{R}^m \). (This is a straightforward extension of the definition of a sequence of real numbers.) It’s fairly simple to extend to \( \mathbb{R}^m \) ideas about convergence of sequences. The following definition is a straightforward extension of the definition for convergence for sequences of real numbers.

**Definition 3.2 (Convergence of sequences in \( \mathbb{R}^m \))** Suppose that \((x_n)\) is a sequence of points of \( \mathbb{R}^m \). Then we say that the sequence has limit \( x \in \mathbb{R}^m \) if for every \( \epsilon > 0 \) there is \( N \) such that if \( n > N \) then \( \|x_n - x\| < \epsilon \). Such a sequence is said to be convergent and to converge towards \( x \).

Equivalently, \( x_n \to x \) as \( n \to \infty \) if
\[
\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty.
\]

The following result says that a sequence converges to a point if and only if it converges in each co-ordinate.

**Theorem 3.2** Suppose \((x_n)\) is a sequence in \( \mathbb{R}^m \) and let \( x_n = (x_{1n}, x_{2n}, \ldots, x_{mn}) \). Then \( x_n \to x = (x_1, \ldots, x_m) \) if and only if, for \( i = 1, \ldots, m \), \( x_{in} \to x_i \) as \( n \to \infty \).

**Proof**
Suppose \( x_n \to x \) and let \( \epsilon > 0 \) be given. Then there is \( N \) such that for \( n > N \),
\[
\|x_n - x\| < \epsilon.
\]
But
\[
\|x_n - x\| = \sqrt{\sum_{i=1}^{m} (x_{in} - x_i)^2} \geq |x_{in} - x_i|,
\]

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for any \( i \) between 1 and \( m \), so for \( n > N \), \( |x_{in} - x_i| < \epsilon \) and hence \( x_{in} \to x_i \). On the other hand, if, for each \( i \), \( |x_{in} - x_i| < \alpha \), then

\[
\|x_n - x\| = \sqrt{\sum_{i=1}^{m} (x_{in} - x_i)^2} < \sqrt{ma^2} = \sqrt{m}\alpha,
\]

so if we let \( \alpha = \epsilon/\sqrt{m} \), we have:

\[
|x_{in} - x_i| < \frac{\epsilon}{\sqrt{m}}, \; (i = 1, 2, \ldots, m) \implies \|x_n - x\| < \epsilon.
\]

If \( x_{in} \to x_i \) for each \( i \), we may take \( \epsilon/\sqrt{m} \) in the definition of limit (in place of \( \epsilon \)) to see that there is some \( N_i \) such that for \( n > N_i \), \( |x_{in} - x_i| < \epsilon/\sqrt{m} \). Let \( N \) be the largest of \( N_1, \ldots, N_m \). Then for \( n > N \), \( |x_{in} - x_i| < \epsilon/\sqrt{m} \) for all \( i \) and hence \( \|x_n - x\| < \epsilon \). This shows that \( x_n \to x \).

\[\square\]

**Example 3.1** Suppose \( x_n = \left( \frac{n}{4n^2 + 2}, \frac{1}{n^2 - 1} \right) \). Then, as \( n \to \infty \), \( x_n \to x = \left( \frac{1/4}{1} \right) \). To see this, we can simply observe that \( \frac{n}{4n^2 + 2} \to \frac{1}{4} \) and \( \frac{n^2}{n^2 - 1} \to 1 \).

Alternatively (though this is more difficult), we could calculate \( \|x_n - x\| \) and check that \( \|x_n - x\| \to 0 \) as \( n \to \infty \).

### 3.2.3 Bolzano-Weierstrass theorem

**Definition 3.3** A sequence in \( \mathbb{R}^m \) is bounded if the set \( \{x_n : n \in \mathbb{N}\} \) is a bounded subset of \( \mathbb{R}^m \).

Thus, a sequence \( (x_n) \) is bounded if there is some number \( M \) such that \( \|x_n\| \leq M \) for all \( n \).

Many of the results for sequences of real numbers extend to sequences in \( \mathbb{R}^m \) for \( m > 1 \). For example, the Bolzano-Weierstrass theorem has the following generalisation.

**Theorem 3.3 (Bolzano-Weierstrass)** Any bounded sequence in \( \mathbb{R}^m \) has a convergent subsequence.

### 3.3 Revision: limits and continuity of functions

\( f : \mathbb{R} \to \mathbb{R} \)

This section acts to remind us briefly of the important ideas of limit and continuity of functions from \( \mathbb{R} \) to \( \mathbb{R} \).
3. Sequences, functions and limits in higher dimensions

3.3.1 Limits of functions \( f : \mathbb{R} \rightarrow \mathbb{R} \)

**Definition 3.4 (Limit of a function at a point)** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function. We say that \( L \) is the limit of \( f(x) \) as \( x \) tends to \( a \), denoted by \( \lim_{x \to a} f(x) = L \), if for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[ 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon. \]

The definition states that if someone gives us any arbitrarily small \( \epsilon \), then there is some neighbourhood of \( a \), \((a - \delta, a + \delta)\), such that any \( x \) in this neighbourhood — other than possibly \( a \) itself — will have \( f(x) \) in the \( \epsilon \)-neighbourhood \((L - \epsilon, L + \epsilon)\) of \( L \). (Note that what happens to the function at \( a \) is irrelevant.)

Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be two functions and \( c \) any real number. Then a new function \((f + g)\) is obtained by defining for each \( x \), \((f + g)(x) = f(x) + g(x)\). Similarly, we may define the functions \(|f|, (cf), (f - g), (f + g), (fg)\) and \((f/g)\), provided \( g(x) \neq 0 \). (For example, \((fg)(x) = f(x)g(x)\). This should not be confused with the composite function \( f(g(x))\).)

**Theorem 3.4** Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be two functions and \( c \) any real number. Suppose that \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \). Then

1. \( \lim_{x \to a} (cf)(x) = cL \)
2. \( \lim_{x \to a} (|f|)(x) = |L| \)
3. \( \lim_{x \to a} (f + g)(x) = L + M \)
4. \( \lim_{x \to a} (f - g)(x) = L - M \)
5. \( \lim_{x \to a} (fg)(x) = LM \)
6. \( \lim_{x \to a} (f/g)(x) = L/M \) provided \( g(x) \neq 0 \) for each \( x \) in some neighbourhood of \( a \).

**Definition 3.5 (One-sided limits)** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function. We say that \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \) from the left, denoted by \( \lim_{x \to a^-} f(x) = L \) if for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[ 0 < a - x < \delta \implies |f(x) - L| < \epsilon. \]

A similar definition applies to limits from the right, denoted \( \lim_{x \to a^+} f(x) = L \).

3.3.2 Continuity of functions \( f : \mathbb{R} \rightarrow \mathbb{R} \)

**Definition 3.6 (Continuity at a point)** A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at the point \( a \) if \( \lim_{x \to a} f(x) \) exists and equals \( f(a) \).

**Definition 3.7 (Continuity on a set)** Suppose \( X \subseteq \mathbb{R} \). A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous on \( X \) if for each \( a \in X \), the limit of \( f(x) \), as \( x \to a \) and \( x \in X \), exists and equals \( f(a) \).

Here is a special case of this definition.
3.4 Limits and continuity of functions $f : \mathbb{R}^n \to \mathbb{R}^m$

Definition 3.8 (Continuity on an interval) A function is continuous on the interval $[a, b]$ if it is continuous at each point in $(a, b)$ and (i) $f(a) = \lim_{x \to a^+} f(x)$ and (ii) $f(b) = \lim_{x \to b^-} f(x)$.

So, to say that $f$ is continuous on $[a, b]$ means that $f$ is continuous at each point in $(a, b)$, and that it is continuous on the left at $b$ and continuous on the right at $a$.

Definition 3.9 (Continuity) A function is continuous if it is continuous at each point $a$ where it is defined.

A function is discontinuous at $a$ if it is not continuous there.

It follows from the results on the algebra of limits that there are ‘heredity’ results for continuity.

Theorem 3.5 (Heredity results for continuity) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions that are continuous at $a \in \mathbb{R}$ and $c$ be any real number. Then $|f|, (cf), (f - g), (f + g), (fg)$ are all continuous at $a$, and $(f/g)$ is continuous provided $g(x) \neq 0$ for any $x$ in some neighbourhood of $a$.

As a corollary, any polynomial $p(x) = \sum_{i=0}^{k} a_i x^i$ is continuous.

Recall that if $f, g$ are functions, then we may define the composite function $f(g(x))$. It turns out that if $g$ is continuous at $a$, and $f$ is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at $a$.

3.4 Limits and continuity of functions $f : \mathbb{R}^n \to \mathbb{R}^m$

We now turn our attention to functions defined on $\mathbb{R}^n$.

3.4.1 Limits of functions $f : \mathbb{R}^n \to \mathbb{R}^m$

Suppose that $f : \mathbb{R} \to \mathbb{R}$. As mentioned above, we say that $f(x)$ tends to $L$ as $x \to a$ if and only if given any $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$ 

This appears to use the fact that one can form the difference between any two real numbers, However, as above, we can interpret $|x - a|$ as the distance between the real numbers $x$ and $a$, in which case the above condition can be restated as

$$0 < \text{distance}(x, a) < \delta \implies \text{distance}(f(x), L) < \epsilon.$$ 

It should be clear from this that the condition doesn’t really use any algebraic properties of $\mathbb{R}$, only ‘distance’ properties. This definition and many of its consequences will remain if we have as domain and codomain $\mathbb{R}^n$ and $\mathbb{R}^m$ for any $m, n \geq 1$.

Definition 3.10 Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, and that $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{L} \in \mathbb{R}^m$. We say that $\mathbf{L}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{a}$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies \|f(\mathbf{x}) - \mathbf{L}\| < \epsilon.$$
and we write $\lim_{x \to a} f(x) = L$.

(Note that we use the same notation, $\| \|$, for the lengths of vectors in both $\mathbb{R}^n$ and $\mathbb{R}^m$.)

The definition can be modified in the obvious way if the function $f$ maps from some subset $A$ of $\mathbb{R}^n$: we simply add the qualification that $x \in A$.

### 3.4.2 Two informative examples

We now give two examples to illustrate some important points about considering limits for functions $f : \mathbb{R}^n \to \mathbb{R}^m$. Two key observations are:

- The limit of $f(x)$ as $x \to a$ exists and equals $L$ only if, no matter how $x$ tends to $a$, the value of the function approaches $L$.

- Consideration of particular approaches of $x$ to $a$ along particular ‘trajectories’ (such as lines) can be used to show that a limit does not exist, but it can never be used to show a limit does exist: that requires a more general argument that assumes nothing about how $x$ approaches $a$.

#### Example 3.2

Suppose $f : \mathbb{R}^2 \setminus \{(0, 0)^T\} \to \mathbb{R}$ is given by

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}.$$

Then $f(x) \to 0$ as $x \to 0 = (0, 0)^T$. To see this, we note that

$$|f(x) - 0| = |f(x)| = \frac{|x_1||x_2|}{\sqrt{x_1^2 + x_2^2}} \leq \frac{(1/2)(x_1^2 + x_2^2)}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{2} \sqrt{x_1^2 + x_2^2} \to 0,$$

where we have used the fact that for any real numbers $a$ and $b$, $2ab \leq a^2 + b^2$. (This follows from $(a - b)^2 \geq 0$.)

When looking at limits for functions from $\mathbb{R}$ to $\mathbb{R}$, we noticed that one can define left and right limits and that these might be different. A counterpart to the idea of left and right limits for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ when $n > 1$ is the idea of the limit along a path.

#### Example 3.3

Suppose $f : \mathbb{R}^2 \setminus \{(0, 0)^T\} \to \mathbb{R}$ is given by

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_2^2 - x_1^2}{x_1^2 + x_2^2}$$

and that $g : \mathbb{R}^2 \setminus \{(0, 0)^T\} \to \mathbb{R}$ is

$$g \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_1 x_2^2}{x_1^2 + x_2^2}.$$
Let’s consider what happens to $f(x)$ as $x$ tends to $0 = (0, 0)^T$ along the line $x_2 = \alpha x_1$; that is, through $x$ of the form $(t, \alpha t)^T$. We have

$$f\left(\begin{pmatrix} t \\ \alpha t \end{pmatrix}\right) = \frac{\alpha^2 t^2 - t^2}{t^2 + \alpha^2 t^2} = \frac{\alpha^2 - 1}{\alpha^2 + 1}.$$ 

So, $f(x)$ approaches different values as $x \to 0$ along different lines. In particular, $f(x)$ does not have a limit as $x \to 0$.

The function $g$ is quite different. If we again investigate what happens as $x \to 0$ along the lines $x_2 = \alpha x_1$, we note that, for all $\alpha$,

$$g\left(\begin{pmatrix} t \\ \alpha t \end{pmatrix}\right) = \frac{\alpha^2 t^3}{t^2 + \alpha^4 t^4} \to 0 \text{ as } t \to 0.$$ 

So, here, the limit as $x$ tends to $0$ along all lines is the same. However, we cannot deduce from this alone that $g(x)$ has a limit as $x \to 0$. For, consider what happens to $g(x)$ as $x \to 0$ along the parabola given by $x_1 = \alpha x_2^2$ (that is, through points $(\alpha t^2, t)$). We have

$$g\left(\begin{pmatrix} \alpha t^2 \\ t \end{pmatrix}\right) = \frac{\alpha \alpha t^4}{\alpha^2 t^4 + t^4} = \frac{\alpha}{\alpha^2 + 1},$$ 

so the limit depends on $\alpha$, and so $g(x)$ has no limit as $x \to 0$.

The two examples just given are very, very important and illustrate why the topic of limits for functions $f: \mathbb{R}^n \to \mathbb{R}^m$ is quite hard when $n > 1$. We repeat the main lessons to be learned. There are so many different ways in which $x$ can approach a given $a$. The limit of $f(x)$ as $x \to a$ exists and equals $L$ only if, no matter how $x$ tends to $a$, the value of the function approaches $L$. Consideration of particular approaches of $x$ to $a$ along particular ‘trajectories’ (such as lines) can be used to show that the limit does not exist (because, for instance, the values along different trajectories tend to different limits). However, to show that the limit of $f$ as $x$ tends to $a$ exists, an argument needs to be given that does not assume any particular way in which $x$ tends towards $a$.

### 3.4.3 Continuity of functions $f: \mathbb{R}^n \to \mathbb{R}^m$

**Definition 3.11 (Continuity of $f: \mathbb{R}^n \to \mathbb{R}^m$)** Suppose that $f: \mathbb{R}^n \to \mathbb{R}^m$ and that $a \in \mathbb{R}^n$. Then, we say that $f$ is **continuous at** $a$ if $\lim_{x \to a} f(x)$ exists and equals $f(a)$. Equivalently, $f$ is continuous at $a$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x - a\| < \delta$ then $\|f(x) - f(a)\| < \epsilon$.

For a subset $X$ of $\mathbb{R}^m$, we say that $f$ is **continuous on** $X$ if for all $a \in X$, the limit of $f(x)$, as $x \to a$, with $x \in a$, exists and equals $f(a)$.

If $f$ is continuous on the whole of $\mathbb{R}^m$ then we simply say that $f$ is **continuous**.

The following result is often useful.
Theorem 3.6 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ and that $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are such that for all $x \in \mathbb{R}^n$, 

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T.$$ 

Then $f$ is continuous at $a \in \mathbb{R}^n$ if and only if $f_1, f_2, \ldots, f_m$ are continuous at $a$.

Note that if $e_1, e_2, \ldots, e_m$ are the standard basis vectors of $\mathbb{R}^m$ (so that $e_i$ has a 1 in position $i$ and all other entries equal to 0), then the functions $f_i$ referred to are given by 

$$f_i(x) = \langle f(x), e_i \rangle = e_i^T f(x),$$

where $\langle a, b \rangle$ denotes the usual inner product (scalar product) on $\mathbb{R}^m$. We shall call the functions $f_i$ the \textit{component functions} of $f$.

A useful observation is that all linear functions are continuous. Recall that a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is one with the property that for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$, 

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Any linear function can be represented in matrix form: that is, there is some $m \times n$ matrix $M$ such that $f(x) = Mx$ for all $x$. In this case, for $1 \leq i \leq m$, 

$$f_i(x) = e_i^T f(x) = e_i^T Mx.$$ 

If we let $m_i$ denote $M^T e_i$, then we see that 

$$f_i(x) = e_i^T Mx = (M^T e_i)^T x = m_i^T x,$$

and so, for any $a \in \mathbb{R}^n$, 

$$\|f_i(x) - f_i(a)\| = \|m_i^T x - m_i^T a\| = \|m_i^T (x - a)\| \leq \|m_i^T\| \|x - a\|.$$ 

As $x \to a$, the right hand side tends to 0, because $\|m_i^T\|$ is just a fixed number. This shows that $f_i$ is continuous, for each $i$, and it follows that $f$ is continuous.

3.4.4 Sequences and continuity

It is possible to describe an alternative approach to the definition of continuity of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. We have the following theorem. (In Abstract Mathematics, we met the version of this that applies to functions $f : \mathbb{R} \to \mathbb{R}$.)

Theorem 3.7 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ and that $a \in \mathbb{R}^n$. Then $f$ is continuous at $a$ if and only if for every sequence $(x_n)$ converging to $a$ we have $f(x_n) \to f(a)$. Therefore $f$ is continuous (on the whole of $\mathbb{R}^n$) if for every convergent sequence $(x_n)$ in $\mathbb{R}^n$, we have $\lim f(x_n) = f(\lim x_n)$.

Proof

Let $(\ast)$ be the statement that for any sequence $(x_n)$ such that $\lim_{n \to \infty} x_n = a$, $\lim_{n \to \infty} f(x_n) = f(a)$.

Suppose first that $f$ is continuous at $a$. We prove that this implies $(\ast)$. Let $(x_n)$ be a sequence of reals converging to $a$. We want to show that $f(x_n) \to f(a)$ as $n \to \infty$, that is, 

$$\forall \epsilon > 0 \ \exists N \ \forall n \geq N \ |f(x_n) - f(a)| < \epsilon.$$ 

(\ast\ast)
To prove this, let $\epsilon > 0$. Choose, according to the definition of continuity, a $\delta > 0$ so that for all $x$, whenever $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Since $x_n \to a$ as $n \to \infty$, there is an $N$ so that $n \geq N$ implies $|x_n - a| < \delta$, which in turn implies $|f(x_n) - f(a)| < \epsilon$. This shows $\text{(<*)}$ as desired.

Conversely, assume that property $(\cdot)$ holds. In order to show continuity, we assume, to the contrary, that the function is discontinuous at $a$. This means that there is an $\epsilon > 0$ so that for all $\delta > 0$ there is an $x$ with $|x - a| < \delta$ but $|f(x) - f(a)| \geq \epsilon$. In particular, for every natural number $n$, letting $\delta = 1/n$, there is a real number $x$, call it $x_n$, with $|x_n - a| < 1/n$ but $|f(x_n) - f(a)| \geq \epsilon$. But then clearly $x_n \to a$ as $n \to \infty$, but we do not have $f(x_n) \to f(a)$ as $n \to \infty$, a contradiction to $(\cdot)$.

\begin{center}
\textbf{Learning outcomes}
\end{center}

At the end of this chapter and the relevant reading, you should be able to:

- state precisely what’s meant by a bounded set in $\mathbb{R}^m$
- state the definition of convergence of a sequence in $\mathbb{R}^m$, and be able to use it
- state, and be able to prove, the result stating that a sequence $(x_n)$ in $\mathbb{R}^m$ converges to $x$ if and only if, for each $i$ between 1 and $m$, the $i$th entries of $x_n$ converge to the $i$th entry of $x$
- state the Bolzanno-Weierstrass theorem (but there is no need to be able to prove it)
- state the definitions of limit and continuity (including continuity on an interval, etc) of a function $f : \mathbb{R} \to \mathbb{R}$
- state the definition of the limit of a function $f : \mathbb{R}^m \to \mathbb{R}^n$
- state the definition of continuity of a function $f : \mathbb{R}^m \to \mathbb{R}^n$
- prove that certain functions are continuous, or not continuous at certain points; and that certain limits do or do not exist
- state the result relating continuity of $f : \mathbb{R}^m \to \mathbb{R}^n$ to continuity of the component functions of $f$
- state, and be able to prove, the characterisation of continuity in terms of sequences

\begin{center}
\textbf{Sample examination questions}
\end{center}

\textbf{Question 3.1}

For $n \in \mathbb{N}$, let the point $x_n$ in $\mathbb{R}^3$ be given by $x_n = (1/n, 1/n, 2/n)$. Calculate $\|x_n\|$ for each $n$, and hence show that $x_n \to 0$ as $n \to \infty$.

\textbf{Question 3.2}

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions, and $c$ a real number. Suppose that $\lim_{x \to c} f(x) = A$ and $\lim_{x \to c} g(x) = B$. Prove, directly from the definitions, that $\lim_{x \to c} (f(x)g(x)) = AB$. Hint: $f(x)g(x) - AB = f(x)(g(x) - B) + (f(x) - A)B$. 

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Question 3.3
Show that \( \lim_{x \to 0} f(x) = 0 \), when \( f(x_1, x_2)^T = \frac{x_1^2 + x_2^2}{|x_1| + |x_2|} \).

Question 3.4
Prove, directly from the definition, that the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = x_1x_2 \) is continuous.

Question 3.5
Does \( \lim_{x \to 0} f(x) \) exist, when \( f(x_1, x_2) = x_1x_2 |x_1| + |x_2| \)?

[You might make use of the fact that, for any \( u, v \), \( u^2 + v^2 \geq 2uv \).]

Question 3.6
Suppose \( f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \) is given by \( f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \frac{x_1^3x_2 - 2x_1^2x_2^3}{x_1^4 + x_2^4} \).
Prove that \( \lim_{x \to 0} f(x) \) does not exist.

Question 3.7
Suppose \( g : \mathbb{R}^2 \to \mathbb{R} \) is the function given by:
\[
\begin{cases} 
  x_1x_2^4 & \text{if } x_1 \geq x_2 \\
  \frac{x_1^2}{x_1^2 + x_2^2} & \text{if } x_1 < x_2 \\
  0 & \text{if } (x, y)^T = 0.
\end{cases}
\]

Prove that \( g \) is not continuous at \( 0 \).

Question 3.8
Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be \( f \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{cases} 
  (x, 0)^T & \text{if } y \leq 0 \\
  (x, x)^T & \text{if } y > 0.
\end{cases} \)
Determine the set of \( a \in \mathbb{R}^2 \) at which \( f \) is continuous.

Question 3.9
Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is the function given by
\[
f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \begin{cases} 
  \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) & \text{if } x_1 \geq x_2 \\
  \left( \begin{array}{c} x_1 \\ 2x_2 - x_1 \end{array} \right) & \text{if } x_1 < x_2.
\end{cases}
\]
Prove that \( f \) is continuous (on all of \( \mathbb{R}^2 \)).
Comments on selected activities

Feedback to activity 3.1
This is an ‘if and only if’ problem. Separate the two halves completely; otherwise only confusion will ensue.

Draw pictures to see why this result is true; if you just launch into a calculation you won’t succeed. So, if the ‘diameter’ $K$ of $B$ is finite, why can there not be points of $B$ arbitrarily far from the origin, and how then can we put some specific upper bound $M$ on the distance from the origin? For the other way round, if all points of $B$ are at distance at most $M$ from the origin, how far can a pair of points of $B$ be from each other? Once you’ve figured out (in both directions) what to prove, the main weapon at your disposal is the triangle inequality.

First suppose that $B$ is bounded, i.e., there is some constant $K$ such that $\|x - y\| \leq K$ for all $x, y \in B$. If $B = \emptyset$ then the result is trivial. If $B \neq \emptyset$, then fix some $y \in B$, and observe that, for all $x \in B$, $\|x\| = \|x - 0\| \leq \|x - y\| + \|y - 0\| \leq K + \|y\|$, by the triangle inequality. Therefore, taking $M = K + \|y\|$, we see that $\|x\| \leq M$ for all $x \in B$.

Now suppose that there is some $M$ such that $\|x\| \leq M$ for all $x \in B$. Take any $x, y$ in $B$, and observe that $\|x - y\| \leq \|x - 0\| + \|0 - y\| = \|x\| + \|y\| \leq M + M = 2M$. So, setting $K = 2M$, we see that $\|x - y\| \leq K$ for all $x, y \in B$, as required.

Sketch answers to or comments on selected questions

Answer to question 3.1
$\|x_n\| = \sqrt{6}/n$. This tends to zero as $n \to \infty$. That is, $\|x_n - 0\| \to 0$ and so $x_n \to 0$.

Answer to question 3.2
We have

$$|f(x)g(x) - AB| = |f(x)(g(x) - B) + (f(x) - A)B|$$

$$\leq |f(x)||g(x) - B| + |f(x) - A||B|.$$ 

Now, because $f(x) \to A$ as $x \to c$, there is (taking $\epsilon = 1$ in the definition) $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ then $|f(x) - A| < 1$, which implies that $|f(x)| < |A| + 1$. Let $\epsilon > 0$. Because $f(x) \to A$ and $g(x) \to B$ as $x \to c$, there are $\delta_2, \delta_3 > 0$ such that

$$0 < |x - c| < \delta_2 \implies |f(x) - A| < \frac{\epsilon}{2(|A| + 1)},$$

$$0 < |x - c| < \delta_3 \implies |g(x) - B| < \frac{\epsilon}{2|B|},$$

supposing that $B \neq 0$ (the proof being easier when $B = 0$ for the second term is not present). Why did I choose these strange choices of ‘$\epsilon$’ in the definition? Well, because if
3. Sequences, functions and limits in higher dimensions

$0 < |x - c| < \delta$, where $\delta = \min(\delta_1, \delta_2, \delta_3)$, then all three inequalities hold and we have

$$|f(x)g(x) - AB| \leq |f(x)||g(x) - B| + |f(x) - A||B|$$

$$< (|A| + 1) \frac{\epsilon}{2(|A| + 1)} + \frac{\epsilon}{2|B||B|}$$

$$= \epsilon,$$

which is what we need.

**Answer to question 3.3**

Note that

$$\left| f \left( \frac{x_1}{x_2} \right) \right| \leq \frac{|x_1|^2}{|x_1| + |x_2|} + \frac{|x_2|^2}{|x_1| + |x_2|}$$

$$\leq \frac{|x_1|^2}{|x_1|} + \frac{|x_2|^2}{|x_2|}$$

$$= |x_1| + |x_2|$$

$$\leq 2\|(x_1, x_2)^T\|$$

$$= 2\|x\|.$$

Therefore, if $\|x\| \to 0$, so does $|f(x)|$, i.e., $\lim_{x \to 0} f(x) = 0$. Do you notice that this proof does not assume any particular ‘type’ of approach of $x$ to $0$? That is, we don’t assume that $x$ approaches $0$ along any particular line, curve, or ‘trajectory’. That’s important. Considering approaches along particular trajectories can sometimes be used to prove that a limit does *not* exist (because you might be able to show that the limit depends on the choice of trajectory). But to prove that a limit exists, the argument given must be trajectory-independent. (Even proving, for example, that the limit exists along all trajectories of the form $x = (t, t^s)$ for all $s$ is not enough, because this does not cover all possible approaches to $0$. There are many other trajectories that aren’t of this form, such as $(t, \sin t)$.)

**Answer to question 3.4**

Fix some $a = (a_1, a_2)^T \in \mathbb{R}^2$ and some $\epsilon > 0$. We need to show there is $\delta > 0$ such that $\|x - a\| < \delta$ implies $|f(x) - f(a)| < \epsilon$; that is, $|x_1x_2 - a_1a_2| < \epsilon$. Now, suppose that $\|x - a\| < \delta$. Then,

$$|x_1x_2 - a_1a_2| \leq |x_1||x_2 - a_2| + |x_1 - a_1||a_2| \leq (|x_1| + |a_2|)\|x - a\|.$$

Now, let’s suppose that $\delta \leq 1$. Then, certainly, $|x_1| \leq 1 + |a_1|$ and hence we have

$$|x_1x_2 - a_1a_2| \leq (1 + |a_1| + |a_2|)\|x - a\|$$

which will be less than $(1 + |a_1| + |a_2|)\delta$ if $\|x - a\| < \delta$. This is at most $\epsilon$ provided $\delta \leq \epsilon/(1 + |a_1| + |a_2|)$. To make this work, we’ve supposed that $\delta < 1$ and we also want $\delta \leq \epsilon/(1 + |a_1| + |a_2|)$. So it suffices to take $\delta = \min(1, \epsilon/(1 + |a_1| + |a_2|))$. (Think about it: this isn’t hard. Two assumptions about how small $\delta$ is make the argument work, so to encapsulate those assumptions as one, just make $\delta$ as small as the smaller of the two.)
3.4. Sketch answers to or comments on selected questions

Answer to question 3.5

We use the fact that for $u, v \geq 0$, $2uv \leq u^2 + v^2$. (This useful inequality follows directly from $(u - v)^2 \geq 0$.) Taking $u = \sqrt{|x_1|}$ and $v = \sqrt{|x_2|}$, we therefore have

$$2|x_1x_2| = 2\sqrt{|x_1||x_2|\sqrt{|x_1||x_2|} \leq \sqrt{|x_1||x_2| (|x_1| + |x_2|)},$$

so, for all $x_1, x_2$,

$$\left| f \left( \frac{x_1}{x_2} \right) \right| = \frac{|x_1||x_2|}{|x_1| + |x_2|} \leq \frac{1}{2} \sqrt{|x_1||x_2|} \to 0, \text{ as } x \to (0, 0)^T.$$

It follows that $f(x) \to 0$ as $x \to 0$. There are other acceptable ways to do this. We could, for instance, note that

$$\frac{x_1x_2}{|x_1| + |x_2|} \leq \frac{1}{2} \frac{x_1^2 + x_2^2}{|x_1| + |x_2|},$$

and then proceed as in Question 3.3.

Answer to question 3.6

Consider $f(t, \alpha t)$. This is

$$\frac{\alpha t^4 - 2\alpha^2 t^4}{t^4 + \alpha^4 t^4} = \frac{\alpha - 2\alpha^2}{1 + \alpha^4}.$$

This shows that as $x \to 0$ along the line with equation $x_2 = \alpha x_1$, $f(x) \to (\alpha - 2\alpha^2)/(1 + \alpha^4)$. (In fact, it equals this value at every point of the line, other than the origin, where the function is not defined.) But this value depends on $\alpha$. Because it is different for different values of $\alpha$, $f(x)$ does not approach a limit as $x \to 0$.

Answer to question 3.7

Consider what happens to $g$ as we tend to 0 along the path with equation $x_1 = t^4, x_2 = \alpha t$. We have

$$g \left( \frac{t^4}{\alpha t} \right) = \frac{\alpha^4 t^8}{t^8 + \alpha^8 t^8} = \frac{\alpha^4}{1 + \alpha^8}.$$

This limit depends on the value of $\alpha$, and in particular, does not equal $g(0) = 0$ if $\alpha \neq 0$. The function is therefore not continuous at 0. (In fact, the limit as $x \to 0$ does not even exist; for, if it did, the limit along any path heading towards 0 would be independent of the precise path taken.

Answer to question 3.8

Before tackling the question, let’s think about continuity for the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 1$ for $x > 0$ and $g(x) = 0$ for $x \leq 0$. At any point $a > 0$, there is some neighbourhood $(a - \delta, a + \delta)$ of $a$ where the function is equal to 1 throughout the neighbourhood, so $g$ is certainly continuous at $a$. Similarly for $a < 0$. The problem comes at the interface between the two definitions, namely at the point $a = 0$, where of course $g$ is not continuous. Our two dimensional problem is broadly similar, but here
the interface consists of a line, namely the $x$-axis, and different things might (and in fact do) happen at different points on the line.

We consider a general $a = (a, b) \in \mathbb{R}^2$. Let’s get the easy cases out the way first.

Suppose first that $b > 0$. Then $f(a) = (a, a)^T$. For all $x = (x, y)^T$ sufficiently close to $a$, we must have $y > 0$. (This is clear: suppose that $0 < \|x - a\| < b$. Then $|y - b| \leq \|x - a\| < b$, so $y - b > -b$, from which we obtain $y > 0$.) For such $x$, $f(x) = (x, x)^T$, and this tends to $f(a) = (a, a)$ as $x \to a$, and hence as $x \to a$. So $f$ is continuous at $a$ if $b > 0$. If $b < 0$, then $f(a) = (b, 0)^T$ and, for all $x$ sufficiently close to $a$, $y < 0$, and so $f(x) = (x, 0)^T \to (b, 0)^T = f(a)$ as $x \to a$. So $f$ is continuous at $a$ if $b < 0$.

Suppose now that $b = 0$. Then $f(a) = f(0, 0)^T = (a, 0)^T$. The idea now is that there are points arbitrarily close to $a$ where $f$ takes values close to $(a, a)^T$. That suggests that $f$ is discontinuous at $a$, except in the case $a = 0$.

If $a = 0$, then $f(a) = (0, 0)^T$ and, for $x = (x, y)$, we have either $f(x) = (x, 0)^T$ or $f(x) = (x, x)^T$, both of which tend to $(0, 0)^T$ as $x \to 0$ (and hence as $x \to a = (0, 0)^T$).

So $f$ is continuous at $(0, 0)$. However, if $a \neq 0$, then $f$ is not continuous at $a$. To see this, note that no matter how small $\epsilon > 0$ is, there are $x = (x, y)$ such that $\|x - a\| < \epsilon$ and $y > 0$, so that $f(x) = (x, x)^T$. As $x \to a$, this tends to $(a, a)^T$, which does not equal $f(a) = (a, 0)^T$.

To sum up, $f$ is continuous on $\mathbb{R}^2 \setminus \{(a, 0)^T : a \neq 0\}$; that is, at all points except those of the form $(a, 0)^T$ where $a \neq 0$.

**Answer to question 3.9**

Consider a general $a = (a, b) \in \mathbb{R}^2$.

Suppose first that $a > b$. Then $f(a) = (a, b)^T$. Since $a > b$, it is the case that for all $x = (x, y)^T$ sufficiently close to $a$, $x > y$. (Convince yourself of this: it follows from the fact that for $x$ close to $a$, $x$ is close to $a$ and $y$ is close to $b$, because $|x - a| \leq \|x - a\|$ and $|x - b| \leq \|x - a\|$. Explicitly, if we let $\delta = (a - b)/2 > 0$, then $\|x - a\| < \delta$ implies $|x - a| < \delta$ and $|x - b| < \delta$, so that $x > a - \delta = (a + b)/2$ and $y < b + \delta = (a + b)/2$, so $x > y$.) For $x$ sufficiently close to $a$, therefore, $f(x) = (x, y)^T$. As $x \to a$, $(x, y)^T \to a = f(a)$, so $f$ is continuous at $a$.

If, now, $a < b$, we have, for $x$ sufficiently close to $a$, $x < y$ and $f(x) = (x, 2y - x)^T$.

Since $(x, 2y - x)^T \to (a, 2b - a) = f(a)$ as $x \to a$, $f$ is continuous at $a$.

The only remaining points $a = (a, b)$ are those in which $a = b$. In this case, $f(a) = (a, a)$. For $x = (x, y)^T \in \mathbb{R}^2$, either $f(x) = (x, y)^T$ or $f(x) = (x, 2y - x)^T$. As $x \to (a, a)^T$, we have $(x, y)^T \to (a, a)$ and $(x, 2y - x)^T \to (a, a)$. So,$\lim_{x \to a} f(x) = (a, a)^T = f(a)$, and $f$ is also continuous at these points.

Therefore, $f$ is continuous on all of $\mathbb{R}^2$. 